On[6,4] Error Correcting Codes over GF(7)

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Abstract— **In this paper we investigate the existence, equivalence and some other features of $[6,4]$ error correcting codes over $GF(7)$.**

Keywords- Linear code, generator matrix, equivalent code.

I. Introduction

Throughout this paper, we assume that the alphabet F is the Galois field $GF(q)$, where q is the prime power. A linear code over $GF(q)$ is just a subspace of F^n, the space of all $n-$tuples with components from F. An $[n,k]$ linear code over $GF(q)$ is a $k-$dimensional subspace of F^n. Thus a subset C of F^n is a linear code if and only if (1) $u+v \in C$, for all $u,v \in C$, and (2) $au \in C$, for all $u \in C$, $a \in F$. Since a linear code is a vector sub-space it can be given by a basis. The matrix whose rows are the basis vectors is called a generator matrix. For an acquaintance with coding theory at a basic level the reader may please consult [1,2,3].

A very important concept in coding is the weight of a vector v. By definition, this is the number of non-zero components v has and is denoted by $wt(v)$ The minimum weight of a code, denoted by d is the weight of a non-zero vector of smallest weight in the code. A well-known theorem says that if d is the minimum weight of a code , then C can correct $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ or fewer errors, and conversely. A $[n,k]$ linear code with minimum weight d is often called a $[n,k,d]$ code. In this paper, we intend to explore the [6,4] error-correcting linear codes over $GF(7)$.

Two linear codes over $GF(q)$ are called equivalent if one can be obtained from the other by a combination of operations of the following types:

a) permutation of the positions of the code;
b) multiplication of the symbols appearing in a fixed position by a non-zero scalar.

It is well known [2] that two $k \times n$ matrices generate equivalent linear $[n,k]$ codes over $GF(q)$ if one matrix can be obtained from the other by a sequence of operations of the following types.

1) permutation of the rows;
2) multiplication of a row by a non-zero scalar;
3) addition of a scalar multiple of one row to another;
4) permutation of the columns;
5) multiplication of any column by a non-zero scalar.

It is also worth knowing [2] that if G is a generator matrix of a $[n,k]$ code, then by performing operations of types (1), (2), (3), (4) and (5), G can be transformed to standard form $[I_k \mid A]$, where I_k is the $k \times k$ identity matrix, A is the $k \times (n-k)$ matrix. In this paper, we intend to explore the $[6,4]$ error-correcting linear codes over $GF(7)$ up to equivalence.

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II. Nonexistence of a [6,4] Error Correcting Linear Code over $GF(p)$ if $p \le 4$

In this section, we will show that there exist no [6,4] error correcting code over fields of order 2, 3 or 4.

Theorem (2.1) There exists no [6,4] one error correcting binary, ternary or quaternary code.

Proof. Let M be a generator matrix of a $[6,4]$ code C. Then

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 1 & 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & 1 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & 1 & a_{41} & a_{34} \end{bmatrix}$$

where $a_{ij} \in GF(p)$ for each i, j and p, $1 \le i \le 4$, $1 \le j \le 2$, $1 \le p \le 4$.

If the code is to be error correcting, the minimum weight d should be at least 3. Hence $a_{ij} \ne 0$ for each i and j, $1 \le i \le 4$, $1 \le j \le 2$. One then obtains the following equivalence diagram where r_i and c_i denote the i^{th} row and i^{th} column respectively.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 1 & 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & 1 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & 1 & a_{41} & a_{42} \end{bmatrix} \xrightarrow{a_{11}^{-1}r_1, a_{21}^{-1}r_2, a_{31}^{-1}r_3, a_{41}^{-1}r_4}$$

$$\begin{bmatrix} a_{11}^{-1} & 0 & 0 & 0 & 1 & a_{11}^{-1}a_{12} \\ 0 & a_{21}^{-1} & 0 & 0 & 1 & a_{21}^{-1}a_{22} \\ 0 & 0 & a_{31}^{-1} & 0 & 1 & a_{31}^{-1}a_{32} \\ 0 & 0 & 0 & a_{41}^{-1} & 1 & a_{41}^{-1}a_{42} \end{bmatrix} \xrightarrow{a_{11}c_1, a_{21}c_2, a_{31}c_3, a_{41}c_4}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a_{11}^{-1}a_{12} \\ 0 & 1 & 0 & 0 & 1 & a_{21}^{-1}a_{22} \\ 0 & 0 & 1 & 0 & 1 & a_{31}^{-1}a_{32} \\ 0 & 0 & 0 & 1 & 1 & a_{41}^{-1}a_{42} \end{bmatrix} \xrightarrow{a=a_{11}^{-1}a_{12}, b=a_{21}^{-1}a_{13}, c=a_{31}^{-1}a_{14}, d=a_{41}^{-1}a_{42}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a \\ 0 & 1 & 0 & 0 & 1 & b \\ 0 & 0 & 1 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 1 & d \end{bmatrix} \xrightarrow{a^{-1}c_6} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & a^{-1}b \\ 0 & 0 & 1 & 0 & 1 & a^{-1}c \\ 0 & 0 & 0 & 1 & 1 & a^{-1}d \end{bmatrix}$$

$$\xrightarrow{x=a^{-1}b, y=a^{-1}c, z=a^{-1}d} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & 0 & 1 & y \\ 0 & 0 & 0 & 1 & 1 & z \end{bmatrix} = G.$$

If code C is to be error correcting, all four of $1, x, y, z$ must be distinct as otherwise using linear combinations of rows of G, we can easily show that code C contains vector of weight two. Now all four of $1, x, y, z$ can not be distinct if $p \le 4$. Thus there exists no $[6,4]$ one error correcting binary, ternary or quaternary code. ■.

III. Existence of a [6, 4] Error Correcting Linear Code over GF(p) if $p \ge 5$

By singleton bound $d \le n - k + 1$ for an $[n,k,d]$ code. If $d = n - k + 1$, we call the code a maximum distance separable code or MDS code for short. Hence for a $[6,4]$ code, $d = 3$ is the maximum minimum weight that is attainable. On the other hand, to be 1 error correcting, the minimum weight of a linear code should be at least 3. Hence an 1 error correcting $[6,4]$ code, if it exists, has to be a $[6,4,3]$ MDS code.The next theorem shows that there do always exist an $1-$ error correcting $[6,4]$ code over $GF(p)$ where $p \ge 5$.

Theorem (3.1). Let $GF(p)$ be a field of order p where $p \ge 5$. Then there do always exist a $[6,4]$ error-correcting code over $GF(p)$.

Proof. Let M be a generator matrix of a $[6,4]$ code over $GF(p)$, $p \ge 5$. Then

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 1 & 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & 1 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & 1 & a_{41} & a_{42} \end{bmatrix}$$

where $a_{ij} \in GF(p)$ for each i and j, $1 \le i \le 5, 1 \le j \le 2$. Using the equivalence diagram as in Theorem 2.1 above, we get that M is equivalent to

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & 0 & 1 & y \\ 0 & 0 & 0 & 1 & 1 & z \end{bmatrix}$$

Since $p \ge 5$, exist nonzero $x, y, z \in GF(p)$ such that $1, x$, y and z are all distinct. Then no two columns of

$$H = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 & 0 \\ -1 & -x & -y & -z & 0 & 1 \end{bmatrix}$$

are dependent and exist 3 columns of H

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which are dependent. Hence the minimum weight of the code generated by G or M is 3. ■

Next we show that all the $1-$ error correcting $[6,4]$ codes over $GF(7)$ are equivalent.

Theorem (3.2) An $1-$ error correcting $[6,4]$ code over $GF(7)$ is equivalent to the code with the following generator matrix $\overline{G}$ where

$$\overline{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}.$$

Proof. Let M be a generator matrix of a $[6,4]$ error-correcting code C over $GF(7)$. Then by our earlier discussion in Theorem (2.1), M must be equivalent to

$$G=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&1&0&1&y\\0&0&0&1&1&z\end{bmatrix},$$

where $1,x,y,z$ are all nonzero, distinct and belong to $\{2,3,4,5,6\}$. Notice that there are 6 permutations of x,y and z namely, xyz, xzy, yxz, yzx, zxy and zyx and each yields a matrix as follows:

$$\overline{G_1}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&1&0&1&y\\0&0&0&1&1&z\end{bmatrix},\ \overline{G_2}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&1&0&1&z\\0&0&0&1&1&y\end{bmatrix},\overline{G_3}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&y\\0&0&1&0&1&x\\0&0&0&1&1&z\end{bmatrix},$$

$$\overline{G_4}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&y\\0&0&1&0&1&z\\0&0&0&1&1&x\end{bmatrix},\ \overline{G_5}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&z\\0&0&1&0&1&x\\0&0&0&1&1&y\end{bmatrix} \text{ and}$$

$$\overline{G_6}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&z\\0&0&1&0&1&y\\0&0&0&1&1&x\end{bmatrix}.$$

Below we show that $\overline{G_2}$ is equivalent to $\overline{G_1}$.

$$\overline{G_2}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&1&0&1&z\\0&0&0&1&1&y\end{bmatrix}\xrightarrow{swap(r_3,r_4)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&0&1&1&y\\0&0&1&0&1&z\end{bmatrix}$$

$$\xrightarrow{swap(c_3,c_4)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&x\\0&0&1&0&1&y\\0&0&0&1&1&z\end{bmatrix}=\overline{G_1}.$$

Using similar transpositions of rows and columns, one can show that the remaining $\overline{G}$ s are also equivalent to $\overline{G_1}$. Notice that $\overline{G}=\overline{G_1}$.

Next we notice that x, y and z can be chosen from $\{2,3,4,5,6\}$ in

$$\binom{5}{3}=10$$

ways and they are as follows:

$$\begin{bmatrix}2\\3\\4\end{bmatrix},\begin{bmatrix}2\\3\\5\end{bmatrix},\begin{bmatrix}2\\3\\6\end{bmatrix},\begin{bmatrix}2\\4\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix},\begin{bmatrix}2\\5\\6\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix},\begin{bmatrix}3\\4\\6\end{bmatrix},\begin{bmatrix}3\\5\\6\end{bmatrix} \text{ and } \begin{bmatrix}4\\5\\6\end{bmatrix}.$$

Each of the ten combinations above could yield a generator matrix of a $[6,4]$ code over $GF(7)$, namely

$$G_1=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&4\end{bmatrix},G_2=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&5\end{bmatrix},$$

$$G_3=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&6\end{bmatrix},$$

$$G_4=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&4\\0&0&0&1&1&5\end{bmatrix},G_5=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&4\\0&0&0&1&1&6\end{bmatrix},$$

$$G_6=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix},$$

$$G_7=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&5\end{bmatrix},G_8=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&6\end{bmatrix},$$

$$G_9=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix},G_{10}=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&4\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix}.$$

We will now show that the codes generated by these ten matrices are equivalent. Towards that goal we produce the transitional diagrams below by applying the equivalence operations $(1), (2), (3), (4)$ and (5), mentioned in the Introduction section above. Notice that r_i and c_i denote the i^{th} row and i^{th} column respectively, λr_i and λc_i denote the multiplication of i^{th} row and i^{th} column respectively, $swap(r_i, r_j)$ and $swap(c_i, c_j)$ denote the permutations of i^{th} and j^{th} rows and columns respectively. Finaly, $r_i = r_i + \lambda r_j$ denotes the addition of a scalar multiple of one row, namely λr_j, to another, namely r_i.

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 6 \end{bmatrix} \xrightarrow{r_i = r_i - r_2, i \neq 2}$$

$$\begin{bmatrix} 1 & 6 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 6 & 1 & 0 & 0 & 1 \\ 0 & 6 & 0 & 1 & 0 & 4 \end{bmatrix} \xrightarrow{r_2 = 6r_2} \begin{bmatrix} 1 & 6 & 0 & 0 & 0 & 6 \\ 0 & 6 & 0 & 0 & 6 & 5 \\ 0 & 6 & 1 & 0 & 0 & 1 \\ 0 & 6 & 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{6c_2, 6c_5} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 4 \end{bmatrix} \xrightarrow{swap(c_2, c_5)}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix} = P$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{3c_6} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(r_1, r_2)} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(r_2, r_3)} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(c_1, c_2)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(c_2, c_3)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(c_2, c_3)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix} = G_7$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{2c_6} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{swap(r_1, r_4)} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{swap(r_2, r_3)} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} = G_2$$

$$P=\begin{bmatrix}1&0&0&0&1&6\\0&1&0&0&1&5\\0&0&1&0&1&1\\0&0&0&1&1&4\end{bmatrix}\xrightarrow{swap(r_1,r_4)}\begin{bmatrix}0&0&0&1&1&4\\0&1&0&0&1&5\\0&0&1&0&1&1\\1&0&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_3)}\begin{bmatrix}0&0&0&1&1&4\\0&0&1&0&1&1\\0&1&0&0&1&5\\1&0&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_2)}\begin{bmatrix}0&0&1&0&1&1\\0&0&0&1&1&4\\0&1&0&0&1&5\\1&0&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_4)}\begin{bmatrix}0&0&1&0&1&1\\1&0&0&0&1&4\\0&1&0&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)}\begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&4\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&4\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix}=G_{10}$$

Thus G_3 is equivalent to G_7, G_2 and G_{10}.

$$G_3=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&6\end{bmatrix}\xrightarrow{r_i=r_i-r_3, i\neq 3}$$

$$\begin{bmatrix}1&0&6&0&0&5\\0&1&6&0&0&6\\0&0&1&0&1&3\\0&0&6&1&0&3\end{bmatrix}\xrightarrow{6r_3}\begin{bmatrix}1&0&6&0&0&5\\0&1&6&0&0&6\\0&0&6&0&6&4\\0&0&6&1&0&3\end{bmatrix}$$

$$\xrightarrow{6c_3,6c_5}\begin{bmatrix}1&0&1&0&0&5\\0&1&1&0&0&6\\0&0&1&0&1&4\\0&0&1&1&0&3\end{bmatrix}$$

$$\xrightarrow{swap(c_3,c_5)}\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&6\\0&0&1&0&1&4\\0&0&0&1&1&3\end{bmatrix}=Q$$

$$Q=\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&6\\0&0&1&0&1&4\\0&0&0&1&1&3\end{bmatrix}\xrightarrow{3c_6}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&4\\0&0&1&0&1&5\\0&0&0&1&1&2\end{bmatrix}$$

$$\xrightarrow{swap(r_3,r_4)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&4\\0&0&0&1&1&2\\0&0&1&0&1&5\end{bmatrix}\xrightarrow{swap(r_2,r_3)}$$

$$\begin{bmatrix}1&0&0&0&1&1\\0&0&0&1&1&2\\0&1&0&0&1&4\\0&0&1&0&1&5\end{bmatrix}\xrightarrow{swap(c_3,c_4)}\begin{bmatrix}1&0&0&0&1&1\\0&0&1&0&1&2\\0&1&0&0&1&4\\0&0&0&1&1&5\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&4\\0&0&0&1&1&5\end{bmatrix}=G_4$$

$$Q=\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&6\\0&0&1&0&1&4\\0&0&0&1&1&3\end{bmatrix}\xrightarrow{6c_6}\begin{bmatrix}1&0&0&0&1&2\\0&1&0&0&1&1\\0&0&1&0&1&3\\0&0&0&1&1&4\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_2)}\begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&4\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&3\\0&0&0&1&1&4\end{bmatrix}=G_1$$

$$Q=\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&6\\0&0&1&0&1&4\\0&0&0&1&1&3\end{bmatrix}\xrightarrow{2c_6}\begin{bmatrix}1&0&0&0&1&3\\0&1&0&0&1&5\\0&0&1&0&1&1\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_3)}\begin{bmatrix}1&0&0&0&1&3\\0&0&1&0&1&1\\0&1&0&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_2)}\begin{bmatrix}0&0&1&0&1&1\\1&0&0&0&1&3\\0&1&0&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)}\begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&3\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix}=G_9$$

$$Q=\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&6\\0&0&1&0&1&4\\0&0&0&1&1&3\end{bmatrix}\xrightarrow{5c_3}\begin{bmatrix}1&0&0&0&1&4\\0&1&0&0&1&2\\0&0&1&0&1&6\\0&0&0&1&1&1\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_4)}\begin{bmatrix}0&0&0&1&1&1\\0&1&0&0&1&2\\0&0&1&0&1&6\\1&0&0&0&1&4\end{bmatrix}$$

$$\xrightarrow{swap(r_3,r_4)}\begin{bmatrix}0&0&0&1&1&1\\0&1&0&0&1&2\\1&0&0&0&1&4\\0&0&1&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_3,c_4)}\begin{bmatrix}0&0&1&0&1&1\\0&1&0&0&1&2\\1&0&0&0&1&4\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_3)}\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&4\\0&0&0&1&1&6\end{bmatrix}=G_5$$

Thus G_3 is equivalent to G_1, G_4, G_5 and G_9.

$$G_8=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&6\end{bmatrix}\xrightarrow{5c_6}\begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&1\\0&0&1&0&1&6\\0&0&0&1&1&2\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_2)}\begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&5\\0&0&1&0&1&6\\0&0&0&1&1&2\end{bmatrix}$$

$$\xrightarrow{swap(r_3,r_4)}\begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&5\\0&0&0&1&1&2\\0&0&1&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_3)}\begin{bmatrix}0&1&0&0&1&1\\0&0&0&1&1&2\\1&0&0&0&1&5\\0&0&1&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_3,c_4)}\begin{bmatrix}0&1&0&0&1&1\\0&0&1&0&1&2\\1&0&0&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)}\begin{bmatrix}1&0&0&0&1&1\\0&0&1&0&1&2\\0&1&0&0&1&5\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)} \begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix} = G_6$$

$$G_8 = \begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&6\end{bmatrix} \xrightarrow{r_i=r_i-r_2, i\neq 2}$$

$$\begin{bmatrix}1&6&0&0&0&5\\0&1&0&0&1&3\\0&6&1&0&0&1\\0&6&0&1&0&3\end{bmatrix} \xrightarrow{6r_2}$$

$$\begin{bmatrix}1&6&0&0&0&5\\0&6&0&0&6&4\\0&6&1&0&0&1\\0&6&0&1&0&3\end{bmatrix} \xrightarrow{6c_2,6c_5}$$

$$\begin{bmatrix}1&1&0&0&0&5\\0&1&0&0&1&4\\0&1&1&0&0&1\\0&1&0&1&0&3\end{bmatrix} \xrightarrow{swap(c_2,c_5)} \begin{bmatrix}1&0&0&0&1&5\\0&1&0&0&1&4\\0&0&1&0&1&1\\0&0&0&1&1&3\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_4)} \begin{bmatrix}0&0&0&1&1&3\\0&1&0&0&1&4\\0&0&1&0&1&1\\1&0&0&0&1&5\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_3)} \begin{bmatrix}0&0&0&1&1&3\\0&0&1&0&1&1\\0&1&0&0&1&4\\1&0&0&0&1&5\end{bmatrix}$$

$$\xrightarrow{swap(r_{11},r_2)} \begin{bmatrix}0&0&1&0&1&1\\0&0&0&1&1&3\\0&1&0&0&1&4\\1&0&0&0&1&5\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_4)} \begin{bmatrix}0&0&1&0&1&1\\1&0&0&0&1&3\\0&1&0&0&1&4\\0&0&0&1&1&5\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)} \begin{bmatrix}0&1&0&0&1&1\\1&0&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&5\end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)} \begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&3\\0&0&1&0&1&4\\0&0&0&1&1&5\end{bmatrix} = G_7$$

Since G_8 is equivalent to G_7 and G_7 is equivalent to G_3, we obtain the equivalence between G_8 and G_3.

Finally, we show that G_6 and G_8 are equivalent.

$$G_6 = \begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&2\\0&0&1&0&1&5\\0&0&0&1&1&6\end{bmatrix} \xrightarrow{3c_6} \begin{bmatrix}1&0&0&0&1&3\\0&1&0&0&1&6\\0&0&1&0&1&1\\0&0&0&1&1&4\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_4)} \begin{bmatrix}1&0&0&0&1&3\\0&0&0&1&1&4\\0&0&1&0&1&1\\0&1&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_2,r_3)} \begin{bmatrix}1&0&0&0&1&3\\0&0&1&0&1&1\\0&0&0&1&1&4\\0&1&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(r_1,r_2)} \begin{bmatrix}0&0&1&0&1&1\\1&0&0&0&1&3\\0&0&0&1&1&4\\0&1&0&0&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_4)} \begin{bmatrix}0&0&1&0&1&1\\1&0&0&0&1&3\\0&1&0&0&1&4\\0&0&0&1&1&6\end{bmatrix}$$

$$\xrightarrow{swap(c_2,c_3)} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 6 \end{bmatrix}$$

$$\xrightarrow{swap(c_1,c_2)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 6 \end{bmatrix} = G_8 . \blacksquare$$

IV. Weight Distribution of a $[6,4]$ Linear Code over $GF(7)$

We begin with the following theorem [3].

Theorem (4.1) Let C be a $[n,k,d]$ MDS code over $GF(q)$ with $d = n-k+1$. Then $A_0 = 1,\ A_i = 0,$ $1 \le i < d$ and

$$A_i = \binom{n}{i}\sum_{j=0}^{i-d}(-1)^j\binom{i}{j}(q^{i+1-d-j}-1),\ d \le i \le n$$

where A_i is the number of code-words of weight i.

Applying this theorem on a [6,4] code C we obtain, $A_0 = 1,$

$A_1 = A_2 = 0,$

$$A_3 = \binom{6}{3}(-1)^0\binom{3}{0}(7-1) = 120$$

$$A_4 = \binom{6}{4}\sum_{j=0}^{1}(-1)^j\binom{4}{j}(7^{2-j}-1)$$

$$= 15\left[(-1)^0\binom{4}{0}(48) + (-1)^1\binom{4}{1}(6)\right] = 15(48-24)$$

$$= 360$$

$$A_5 = \binom{6}{5}\sum_{j=0}^{2}(-1)^j\binom{5}{j}(7^{3-j}-1)$$

$$= 6[(7^3-1) - 5\cdot 48 + 10\cdot 6] = 6(342-240+60)$$

$$= 6\cdot 162 = 972$$

$$A_6 = \binom{6}{6}\sum_{j=0}^{3}(-1)^j\binom{6}{j}(7^{4-j}-1)$$

$$= [(7^4-1) - 6\cdot(7^3-1) + 15(7^2-1) - 20(7-1)]$$

$$= 2400 - 2052 + 720 - 120$$

$$= 948$$

It is well-known [1] that if C is an MDS code, so is $C^{\perp}$. Hence the minimum distance of $C^{\perp}$ is $6-2+1=5$. Then by Theorem (4.1) above, $A_0 = 1,\ A_1 = A_2 = A_3 = A_4 = 0$,

$$A_5 = \binom{6}{5}(-1)^0\binom{6}{0}(7-1) = 36 \text{ and}$$

$$A_6 = \binom{6}{6}\sum_{j=0}^{1}(-1)^j\binom{6}{j}(7^{2-j}-1) = 48 - 36 = 12 .$$

Thus we have the following theorem.

Theorem (4.2). A $[6,4]$ error correcting code C over $GF(7)$ has the following weight distribution.

Weight	Number of Words
0	1
3	120
4	360
5	972
6	948

On the other hand, the 2 error correcting $[6,2,5]$ code $C^{\perp}$ has the following weight distribution.

Weight	Number of Words
0	1
5	36
6	12

References

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