A Global Convergence Algorithm for Management Equilibrium Model

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Abstract—**In this paper, we consider the extended linear complementarity problem (ELCP) in management equilibrium modeling. To this end, we first develop some equivalent reformulations of the problem. Based on this, we propose an iterative method for solving the ELCP, its global convergence is proved under mild conditions. Furthermore, we also prove that the method has R-linear convergence rate under suitable conditions.**

***Keywords-** Management equilibrium model; algorithm; global convergence ; R -linear convergence rate; extended linear complementarity problem*

I. INTRODUCTION

Let mappings $F(x)=Mx+p$, $G(x)=Nx+q$, the extended linear complementarity problem, abbreviated as ELCP, is to find vector $x^*\in R^n$ such that

$$\begin{aligned} &F(x^*)\geq 0,\quad G(x^*)\geq 0,\quad F(x^*)^\top G(x^*)=0,\\ &Ax^*+a\leq 0,\quad Bx^*+b=0, \end{aligned} \tag{1}$$

where $M,N\in R^{m\times n}, p,q\in R^m, A\in R^{s\times n}, B\in R^{t\times n}, a\in R^s, b\in R^t$. We assume that the solution set of the ELCP is nonempty throughout this paper, and denote it by X^*.

The ELCP is a direct generalization of the classical linear complementarity problem (LCP) which finds applications in management equilibrium, economics, finance, and operation research ([1,2,3]). For example, the balance of supply and demand is central to all economic systems. This fundamental equation in economics is often described by a complementarity relation between two sets of decision variables([1]). Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables ([2]).

Up to now, the error estimation and existence of the solution for ELCP were fully analyzed ([3-9]). To our knowledge, the issues of numerical methods for solving the problem was also discussed in the literature ([4,5,6,8]), the basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem ([6]), the condition which the nonsingularity of Jacobian at a solution guarantees that the L-M method for ELCP has global convergence ([6]). Sun ([4, 5]) propose some new type of solution methods to solve the ELCP, and show that these algorithms are global convergence under the mapping $G(x)$ is monotone with respect to $F(x)$. This motivates us to consider a new method for the ELCP under mild conditions. So, in this paper, we propose the new iterative method which is different from the algorithms listed above to solve ELCP.

The rest of this paper is organized as follows. In Section 2, we first develop some equivalent reformulations of the ELCP. In Section 3, a new iterative method for solving the ELCP is proposed, and we establish its the global convergence under mild condition. Compared with the existing solution methods in [4, 5], the condition guaranteed for convergence is weaker. Furthermore, under suitable conditions, the R -linear convergence rate analysis of the proposed algorithm is also presented in this paper. Finally, some conclusions are drawn in Section 4.

We end this section with some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean norm of vector in the space is denoted by $\|\cdot\|$. We use $x\geq 0$ to denote a nonnegative vector $x\in R^n$ if there is no confusion.

II. THE EQUIVALENT REFORMULATION OF THE ELCP

In this section, we will establish some equivalent reformulations of the ELCP, and state some well known properties of the projection operator. First, the following result is straightforward.

Proposition 2.1 For $x^*\in R^n$ is a solution of (1) if and only if x^* is a solution of the following problem

$$H(x^*):=\begin{pmatrix} F(x^*)\\ -(Ax^*+a)\\ Bx^*+b\end{pmatrix}\geq 0,\quad Q(x^*):=\begin{pmatrix} G(x^*)\\ 0\\ -(Bx^*+b)\end{pmatrix}\geq 0, \tag{2}$$
$$H(x^*)^T Q(x^*)=0.$$

By (2), a direct computation yields that we can establish the following equivalent formulation of the ELCP.

Proposition 2.2 For $x^* \in R^n$ is a solution of (1) if and only if x^* is a solution of the following problem

$$H(x^*)^T(Q(x)-Q(x^*)) \geq 0, \quad \forall Q(x) \geq 0. \tag{3}$$

By (3), the ELCP can also be equivalently converted into the following problem: find $x^* \in R^n$, such that

$$(x-x^*)^T(\tilde{M}x^* + \hat{q}) \geq 0, \quad \forall x \in \Lambda. \tag{4}$$

where

$\Lambda = \{Nx + q \geq 0, Bx + b \leq 0\}$, $\tilde{M} = N^T M - B^T B$, $\hat{q} = N^T p - B^T b$.

Now, we give the definition of projection operator and some relate properties which will be used in the sequel([11]).

For nonempty closed convex set $\Omega \subset R^n$ and any vector $x \in R^n$, the orthogonal projection of x onto Ω, i.e.,

$$argmin\{\| y - x \| \mid y \in \Omega\},$$

is denoted by $P_\Omega(x)$.

Proposition 2.3 For any $u \in R^n, v \in \Omega$, then

$(i)\langle P_\Omega(u) - u, v - P_\Omega(u)\rangle \geq 0,$

$(ii)\|P_\Omega(u) - P_\Omega(v)\|^2 \leq \|u - v\|^2 - \|(P_\Omega(u) - u) - (P_\Omega(v) - v)\|^2.$

For convenience, we define vector

$$r(u,\rho) := u - P_\Lambda[u - \rho(\tilde{M}u + \hat{q})]$$
$$= u - P_\Lambda[(I - \rho\tilde{M})u - \rho\hat{q}], \quad \rho > 0.$$

Combining Proposition 2.2 with Proposition 2.3, one can prove that (4) is equivalent to the fixed-point problem, this result is due to Noor([10]).

Proposition 2.4 $x^* \in R^n$ is a solution of (1) if and only if $r(x^*, \rho) = 0$ for some $\rho > 0$.

III. Algorithm and Convergence

In this section, we will propose a new iterative method for solving the ELCP, and its the global convergence and R-linear convergence rate were also established under mild condition. Now, we give the following assumptions which are crucial to our method.

Assumption (A1) The mapping $\tilde{F}$ is pseudo-monotone on Λ, i.e., for any $x, y \in \Lambda$, it holds that

$$\langle \tilde{F}(y), x - y\rangle \geq 0 \Rightarrow \langle \tilde{F}(x), x - y\rangle \geq 0,$$

where $\tilde{F}(x) = \tilde{M}x + \hat{q}$.

Certainly, if $\tilde{F}$ is monotone on Λ, then $\tilde{F}$ is pseudo-monotone on Λ. If the mapping $\tilde{F}$ is pseudo-monotone on Λ, then for any $x^* \in X^*$, $\langle \tilde{F}(x), x - x^*\rangle \geq 0, \forall x \in \Lambda.$

Now, we formally state our algorithm.

Algorithm 3.1.

Step 0. Choose any $\sigma, \gamma, \rho \in (0,1), x^0 \in \Lambda, x^{-1} = x^0, k := 0.$

Step 1. For $x^k \in \Lambda$, compute $z^k := P_\Lambda[(I - \rho\tilde{M})x^k - \rho\hat{q}]$. If $\| r(x^k, \rho) \| = 0$, stop; otherwise, compute

$$y^k := (1 - \eta_k)x^k + \eta_k z^k,$$

where $\eta_k = r^{m_k}$ with m_k being the smallest non-negative integer m satisfying

$$\left\langle \tilde{M}x^k + \hat{q}, r(x^k, \rho)\right\rangle \geq (\sigma + \gamma^m \|M\|)\left\|r(x^k, \rho)\right\|^2. \tag{5}$$

Let $x^{k+1} := P_{\Omega_k}(x^k)$ where $\Omega_k = H_k^1 \bigcap H_k^2 \bigcap \Lambda$, and

$$H_k^1 := \{x \in R^n \mid \langle x - y^k, \tilde{M}y^k + \hat{q}\rangle \leq 0\},$$
$$H_k^2 := \{x \in R^n \mid \langle x - x^k, x^{k-1} - x^k\rangle \leq 0\}.$$

In this following, we discuss the feasibility of stepsize rule (5). If Algorithm 3.1 terminates with $\| r(x^k, \rho) \| = 0$, then x^k is a solution of (1). Otherwise, from Proposition 2.3, we have that

$$\left\langle \rho(\tilde{M}x^k + \hat{q}), r(x^k, \rho)\right\rangle - \left\|r(x^k, \rho)\right\|^2$$
$$= \left\langle \rho(\tilde{M}x^k + \hat{q}), x^k - P_\Lambda[x^k - \rho(\tilde{M}x^k + \hat{q})]\right\rangle$$
$$- \left\langle x^k - P_\Lambda[x^k - \rho(\tilde{M}x^k + \hat{q})], x^k - P_\Lambda[x^k - \rho(\tilde{M}x^k + \hat{q})]\right\rangle$$
$$= \left\langle P_\Lambda[x^k - \rho(\tilde{M}x^k + \hat{q})] - (x^k - \rho(\tilde{M}x^k + \hat{q}), x^k - P_\Lambda[x^k - \rho(\tilde{M}x^k + \hat{q})]\right\rangle$$
$$\geq 0.$$

i.e.,

$$\left\langle (\tilde{M}x^k + \hat{q}), r(x^k, \rho)\right\rangle \geq \rho^{-1}\left\|r(x^k, \rho)\right\|^2 > 0. \tag{6}$$

Thus, for large enough number m, so (5) holds by choosing appropriate values of parameters σ, γ, ρ.

The following result says that if the solution set X^* is nonempty, then $X^* \subseteq \Omega_k$ and Ω_k is a nonempty set.

Lemma 3.1 Suppose that Assumption (A1) holds, and the solution set X^* is nonempty. Then, $X^* \subseteq \Omega_k$.

Proof For any $x^* \in X^*$, one has $\langle \tilde{M}x^* + \hat{q}, y^k - x^*\rangle \geq 0$, combining this with Assumption (A1), we obtain that $\langle \tilde{M}y^k + \hat{q}, y^k - x^*\rangle \geq 0, \forall x^* \in X^*$. Thus, $X^* \subseteq H_k^1 \cap \Lambda$.

In this following, we prove that $x^* \subseteq H_k^2$, for any $k \geq 0$ by induction.

Obviously, $x^* \subseteq H_0^2 = R^n$. Suppose that $x^* \subseteq H_k^2$, then

$$X^* \subseteq \Omega_k = H_k^1 \bigcap H_k^2 \bigcap \Lambda.$$

For $x^* \in X^*$, by Proposition 2.3(i), we can obtain

$$\langle x^* - x^{k+1}, x^k - x^{k+1}\rangle = \left\langle x^* - P_{\Omega_k}(x^k), x^k - P_{\Omega_k}(x^k)\right\rangle \leq 0,$$

Thus, $x^* \subseteq H_{k+1}^2$.

This shows that $x^* \subseteq H_k^2$ for any $k \geq 0$, and the desired result follows.

If Algorithm 3.1 terminates at Step 1, then x^k is a solution of this problem. Otherwise, it generates an infinite sequence. Now, we give the global convergence result of Algorithm 3.1.

Theorem3.1. Suppose Assumption (A1) holds, the solution set of (1) is nonempty, and Algorithm 3.1 generates an infinite sequence $\{x^k\}$. Then, the sequence $\{x^k\}$ is bounded and globally converges to a solution of (1).

Proof For any $x^* \in X^*$. Since $x^* \in \Omega_k$ and $x^{k+1} = P_{\Omega_k}(x^k)$, from Proposition 2.3(*ii*), one has

$$\begin{aligned} \| x^{k+1} - x^*\|^2 &= \| P_{\Omega_k}(x^k) - P_{\Omega_k}(x^*)\|^2 \\ &\le \| x^k - x^*\|^2 - \| (P_{\Omega_k}(x^k) - x^k) - (P_{\Omega_k}(x^*) - x^*) \|^2 \\ &= \| x^k - x^*\|^2 - \| P_{\Omega_k}(x^k) - x^k\|^2 \\ &= \| x^k - x^*\|^2 - \| x^{k+1} - x^k\|^2, \end{aligned} \tag{7}$$

By (7), the sequence $\{\|x^k - x^*\|\}$ is non-increasing and bounded. Thus, it converges, and the sequence $\{x^k\}$ is also bounded. From (7) again, we obtain

$$\lim_{k\to\infty} \| x^{k+1} - x^k\|^2 = 0, \tag{8}$$

On the other hand, since

$$P_{H_k^1(x^k)} = x^k - \frac{\langle My^k + \hat{q}, x^k - y^k\rangle}{\|My^k + \hat{q}\|^2}(My^k + \hat{q}).$$

Combining this with $x^{k+1} \in H_k^1$, we have that

$$\begin{aligned} &\| x^k - x^{k+1}\| \ge \| x^k - P_{H_k^1}(x^k)\| \\ &= \left\| \frac{\langle My^k + \hat{q}, x^k - y^k\rangle}{\|My^k + \hat{q}\|^2}(My^k + \hat{q}) \right\| \\ &= \frac{\left\|\langle My^k + \hat{q}, x^k - y^k\rangle\right\|}{\|My^k + \hat{q}\|^2} \left\| My^k + \hat{q} \right\| \\ &= \frac{\left\|\langle M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}, x^k - ((1-\eta_k)x^k + \eta_k z^k)\rangle\right\|}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &= \frac{\left\|\langle (Mx^k + \hat{q}) - \eta_k M(x^k - z^k), \eta_k(x^k - z^k)\rangle\right\|}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &= \frac{\left\|\langle (Mx^k + \hat{q}) - \eta_k M r(x^k,\rho), \eta_k r(x^k,\rho)\rangle\right\|}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &= \frac{\left\|\langle (Mx^k + \hat{q}), \eta_k r(x^k,\rho)\rangle - \langle \eta_k M r(x^k,\rho), \eta_k r(x^k,\rho)\rangle\right\|}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &\ge \frac{\eta_k \left\|\langle (Mx^k + \hat{q}), r(x^k,\rho)\rangle\right\| - \eta_k^2 \left\|M\right\| \left\|r(x^k,\rho)\rangle\right\|^2}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &\ge \frac{\eta_k(\sigma + \eta_k\|M\|)\left\|r(x^k,\rho)\right\|^2 - \eta_k^2\left\|M\right\|\left\|r(x^k,\rho)\rangle\right\|^2}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} \\ &= \frac{\eta_k\sigma\left\|r(x^k,\rho)\right\|^2}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|}. \end{aligned} \tag{9}$$

Combining (9) with (8), we have

$$\lim_{k\to\infty} \frac{\eta_k\sigma\left\|r(x^k,\rho)\right\|^2}{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|} = 0.$$

Since $\{x^k\}$ is bounded, so $\{z^k\}$ is bounded, and thus

$$\{\|M((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|\}$$

is also bounded. It follows that

$$\lim_{k\to\infty} \eta_k \sigma \left\|r(x^k,\rho)\right\|^2 = 0. \tag{10}$$

By (10), we know that

$$\lim_{k\to\infty} \eta_k = 0 \quad \text{or} \quad \lim_{k\to\infty} \left\|r(x^k,\rho)\right\| = 0.$$

For any convergent subsequence $\{x^{k_j}\}$ of $\{x^k\}$, denote its limit by $\bar{x}$, i.e., $\lim_{j\to\infty} x^{k_j} = \bar{x}$. Thus, one has

$$\lim_{j\to\infty} \eta_{k_j} = 0 \quad \text{or} \quad \lim_{j\to\infty} \left\|r(x^{k_j},\rho)\right\| = 0.$$

For the first case, by the choice of η_{k_j} in Algorithm 3.1, we know that

$$\left\langle Mx^{k_j} + \hat{q}, r(x^{k_j},\rho)\right\rangle < (\sigma + \frac{\eta_{k_j}}{\gamma}\|M\|)\left\|r(x^{k_j},\rho)\right\|^2.$$

Combining this, we obtain

$$\lim_{j\to\infty}\left\langle Mx^{k_j} + \hat{q}, r(x^{k_j},\rho)\right\rangle \le \lim_{j\to\infty}(\sigma + \frac{\eta_{k_j}}{\gamma}\|M\|)\left\|r(x^{k_j},\rho)\right\|^2.$$

i.e.,

$$\left\langle M\bar{x} + \hat{q}, r(\bar{x},\rho)\right\rangle \le \sigma\left\|r(\bar{x},\rho)\right\|^2.$$

Using a similar argument of (6), one has

$$\left\langle (M\bar{x} + \hat{q}), r(\bar{x},\rho)\right\rangle \ge \rho^{-1}\left\|r(\bar{x},\rho)\right\|^2$$

with $\sigma, \rho \in (0,1)$, we have $\rho^{-1}\left\|r(\bar{x},\rho)\right\|^2 \le \sigma\left\|r(\bar{x},\rho)\right\|^2$. Thus, we know that $\left\|r(\bar{x},\rho)\right\| = 0$ and thus $\bar{x}$ is a solution of (1).

For the case $\lim_{j\to\infty}\left\|r(x^{k_j},\rho)\right\| = 0$, it is easy to see that the limit point $\bar{x}$ of $\{x^{k_j}\}$ is a solution of (1).

By (7), we know that $\{x^k\}$ is a Fejer sequence with respect to the solution set X^*, the global convergence can easily be obtained.

Next, we discuss the convergence rate of Algorithm 3.1. To this end, we first prove that the sequence $\{\eta_k\}$ generated by Algorithm 3.1 has a positive bound from below.

Lemma 3.2. The sequence $\{\eta_k\}$ generated by Algorithm 3.1 has a positive bound from below.

Proof By the line search procedure for updating η_k, if $\eta_k \le \gamma$, then

$$\left\langle Mx^k + \hat{q}, r(x^k,\rho)\right\rangle < (\sigma + \frac{\eta_k}{\gamma}\|M\|)\left\|r(x^k,\rho)\right\|^2. \tag{11}$$

Using (6), one has

$$\left\langle (Mx^k + \hat{q}), r(x^k,\rho)\right\rangle \ge \rho^{-1}\left\|r(x^k,\rho)\right\|^2. \tag{12}$$

Combining (11) with (12), we obtain

$$\rho^{-1}\left\|r(x^k,\rho)\right\|^2 \le (\sigma + \frac{\eta_k}{\gamma}\|M\|)\left\|r(x^k,\rho)\right\|^2.$$

Thus, $\eta_k \geq \frac{\gamma(\rho^{-1}-\sigma)}{\|M\|}$. From which it follows that, for all k

$$\eta_k \geq \eta := \min\left\{1, \frac{\gamma(\rho^{-1}-\sigma)}{\|M\|}\right\}. \quad (13)$$

In this following, we give the following assumption is needed.

Assumption (A2) There exist positive constants u, δ and $x^* \in X^*$ such that

$$\|x-x^*\| \leq \|r(x,\rho)\|, \quad \forall x \text{ with } \|r(x,\rho)\| \leq \delta.$$

Theorem 3.2. Let Assumptions (A1)-(A2) hold, and parameters $\eta, \sigma, \tau, \beta$ such that $0 < 1 - \frac{(\eta\sigma\beta)^2}{\tau^2} < 1$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 is R -linear convergence to the solution of (1).

Proof From the proof of Theorem 3.1, we know that $\{x^k\}$ and $\{z^k\}$ are bounded sequences, so there exists a positive constant τ such that $\|\hat{M}((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\| \leq \tau$. From (7), (9) and (13), we know that

$$\begin{aligned}
\| x^{k+1} - x^* \|^2 &\leq \| x^k - x^* \|^2 - \| x^{k+1} - x^k \|^2 \\
&\leq \| x^k - x^* \|^2 - \frac{(\eta_k\sigma)^2 \|r(x^k,\rho)\|^4}{\|\hat{M}((1-\eta_k)x^k + \eta_k z^k) + \hat{q}\|^2} \\
&\leq \| x^k - x^* \|^2 - \frac{(\eta_k\sigma)^2 \|r(x^k,\rho)\|^4}{\tau^2} \\
&\leq \| x^k - x^* \|^2 - \frac{(\eta\sigma)^2 \|r(x^k,\rho)\|^4}{\tau^2} \quad (14) \\
&\leq \left[1 - \frac{(\eta\sigma)^2 \|r(x^k,\rho)\|^2}{\tau^2}\right] \| x^k - x^* \|^2 \\
&\leq \left[1 - \frac{(\eta\sigma)^2 \beta^2}{\tau^2}\right] \| x^k - x^* \|^2,
\end{aligned}$$

where the fifth inequality is obtained by Assumption (A2), the last inequality follows from the fact that the sequence $\{x^k\}$ is bounded. Using (14), one has

$$\begin{aligned}
\| x^{k+1} - x^* \|^2 &\leq \left[1 - \frac{(\eta\sigma)^2 \beta^2}{\tau^2}\right] \| x^k - x^* \|^2 \\
&\leq \left[1 - \frac{(\eta\sigma)^2 \beta^2}{\tau^2}\right]^2 \| x^{k-1} - x^* \|^2 \\
&\leq \cdots \\
&\leq \left[1 - \frac{(\eta\sigma)^2 \beta^2}{\tau^2}\right]^{k+1} \| x^0 - x^* \|^2.
\end{aligned}$$

Select the appropriate parameters $\eta, \sigma, \tau, \beta$ such that $0 < 1 - \frac{(\eta\sigma\beta)^2}{\tau^2} < 1$, we have the sequence $\{x^k\}$ generated by the algorithm 3.1 is R -linear convergence to the solution of (1).

IV. CONCLUSIONS

In this paper, we propose a new iterative method for solving ELCP in management equilibrium modeling which has a global convergence and R -linear convergence rate under milder conditions.

According to its limitations, this work has several possible extensions. First, the parameters of Algorithm 3.1 is adjusted dynamically to further enhance the efficiency of the method. Second, Algorithm 3.1 is tailored for ELCP. Therefore, how to extend it to nonlinear complementarity problem, this is an interesting topic for further research.

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