On the Chromatic Spectrum of Uniform $\mathcal{C}$-Hypergraphs

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Abstract—**Motivated by solving the problem proposed by Král, (Electronic J. Combin. 11#R19, 2004), we construct a family of uniform $\mathcal{C}$-hypergraphs whose chromatic spectrum is a linear combination of the Stirling numbers of the second kind.**

Keywords- $\mathcal{C}$ ***-hypergraph; chromatic spectrum; Stirling number of the second kind***

I. INTRODUCTION

A *mixed hypergraph* on a finite set X is a triple $H=(X,C,D)$, where C and D are families of subsets of X. The members of C and D are called C*-edges* and D*-edges*, respectively. A mixed hypergraph with $D=\varnothing$, denoted by $H=(X,C)$, is called a C*-hypergraph*, and a hypergraph with $C=\varnothing$, denoted by $H=(X,D)$, is called a D-hypergraph. A mixed hypergraph is t*-uniform* if every edge contains t vertices.

A *proper* k*-coloring* of a mixed hypergraph H is a mapping from X into a set of k colors such that each C-edge has two vertices with a Common color and each D-edge has two vertices with *Distinct* colors. A coloring may also be viewed as a *partition* (*feasible partition*) of X, where the *color classes* (partition classes) are the sets of vertices assigned to the same color. A *strict* k*-coloring* is a proper k-coloring with k nonempty color classes, and a mixed hypergraph is k*-colorable* if it has a strict k-coloring. The maximum (resp. minimum) number of the colors in a strict coloring of $H=(X,C,D)$ is the *upper chromatic number* (resp. *lower chromatic number*) of H, denoted by $\bar{\chi}(H)$ (resp. $\chi(H)$).

The set of all the values k such that H has a strict k-coloring is called the *feasible set* of H, denoted by $F(H)$. For each k, let r_k denote the number of *feasible partitions* of the vertex set. The vector $R(H)=(r_1,r_2,\ldots,r_{\bar{\chi}})$ is called the *chromatic spectrum* of H, where $\bar{\chi}$ is the upper chromatic number of H. The study of the colorings of mixed hypergraphs has made a lot of progress since its inception [5]. For more information, see [4,6].

It is readily seen that H is a C-hypergraph if and only if $1\in F(H)$. Moreover, if $r_1\neq 0$, then $r_1=1$. For the case $1\notin F(H)$, Jiang et al. [3] proved that for any finite set S of integers greater than 1, there exists a mixed hypergraph H such that $F(H)=S$. Král [2] strengthened this result by showing that prescribing any positive integer r_k, there exists a mixed hypergraph which has precisely r_k strict k-colorings for all $k\in S$. Bujtás and Tuza [1] gave the necessary and sufficient condition for a finite set S of natural numbers being the feasible set of an r-uniform mixed hypergraph. Zhao et al. [8] proved that any vector $R=(0,r_2,\ldots,r_n)$ with $n\geq 2$ and $r_i\geq 0, i=2,\ldots,n$ can be the chromatic spectrum of some 3-uniform bi-hypergraph. Král [2] proposed the following problem: What are necessary and sufficient conditions for a vector $R=(1,r_2,\ldots,r_n)$ to be the chromatic spectrum of a C-hypergraph?

Let $S(n,k)$ be the Stirling number of the second kind, i.e., the number of ways to partition a set of n elements into k nonempty subsets. Assume that $H=(X,C)$ be a t-uniform C-hypergraph and $c=\{C_1,C_2,\ldots,C_n\}$ a strict n-coloring of H. Then for any $t\leq k\leq n$, we can get $S(n,k)$ strict k-colorings of H by union some color classes of c, i.e., $r_k\geq S(n,k)$. In this sense, the n-dimensional vector with the k-th entry being $S(n,k)$ for $t\leq k\leq n$ is a lower bound on the chromatic spectrum of C-hypergraphs with upper chromatic number n. Zhang et al. [7] proved that this lower bound can be attained for 3-uniform C-hypergraphs. In this paper, we generalize it to t-uniform C-hypergraphs and get the following results.

Theorem 1.1 Let $n_1>\cdots>n_s\geq t\geq 3$ be integers. Then for any positive integers $m_1,\ldots,m_s$ and $t_k:=m_1S(n_1,k)+m_2S(n_2,k)+\cdots+m_sS(n_s,k)$, $k=t,\ldots,n_1$, there exists a t-uniform C-hypergraph H with upper chromatic number n_1 and $r_k=t_k$ for each $k\in\{t,\ldots,n_1\}$, where r_k is the k-th entry of the chromatic spectrum of H.

Theorem 1.2 For any integers $n \geq t \geq 3$, there exists a t-uniform C-hypergraph H satisfies that $\bar{\chi}(H) = n$ and

$$R(H) = (1, r_2, \ldots, r_{t-1}, S_2(n,t), \ldots, S_2(n,n)).$$

II. Proof of the Theorems

For any positive integer n, let $[n]$ denote the set $\{1, \ldots, n\}$.

We first give the basic construction of t-uniform C-hypergraphs. For integers $s \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_s \geq t$, let

$$X_{n_1,\ldots,n_s} = \{(x_1,\ldots,x_s) \mid x_j \in [n_j], j \in [s]\},$$
$$C_{n_1,\ldots,n_s} = \{\{\alpha_1,\ldots,\alpha_t\} \mid \alpha_i \in X_{n_1,\ldots,n_s}, i \in [t];$$
$$|\{\alpha_1^j,\ldots,\alpha_r^j\}| \leq t-1, j \in [s]\},$$
$$H_{n_1,\ldots,n_s} = (X_{n_1,\ldots,n_s}, C_{n_1,\ldots,n_s}),$$

where α_i^j is the j-th component of α_i.

Note that for any $i \in [s]$, $c_i^s = \{X_{i1}^s, X_{i2}^s, \ldots, X_{in_i}^s\}$ is a strict n_i-coloring of $H_{n_1,\ldots,n_s}$, where

$$X_{ij}^s = \{(x_1,\ldots,x_{i-1}, j, x_{i+1},\ldots,x_s) \mid x_k \in [n_k], k \in [s] \setminus \{i\}\}, j \in [n_i].$$

Moreover, for any $k \geq t$, we can get $S(n_i,k)$ strict k-colorings of $H_{n_1,\ldots,n_s}$ from c_i^s. Hence, $H_{n_1,\ldots,n_s}$ has at least $S(n_1,k)+\cdots+S(n_s,k)$ strict k-colorings. In the following, we shall prove that $H_{n_1,\ldots,n_s}$ has no other strict k-colorings, that is to say, $r_k = S(n_1,k)+\cdots+S(n_s,k)$ for every $t \leq k \leq n_1$.

Let $k \in \{t,\ldots,n_1\}$, $i \in [s]$ and $c = \{C_1,\ldots,C_k\}$ be one of the $S(n_i,k)$ strict k-colorings of $H_{n_1,\ldots,n_s}$ getting from c_i^s. Note that for each $j \in [n_i]$, there exists an integer in $[k]$, say l_j, such that $X_{ij}^s \subseteq C_{l_j}$. It suffices to prove that for any strict k-coloring $c = \{C_1,\ldots,C_k\}$ of $H_{n_1,\ldots,n_s}$, there exists an $i \in [s]$ such that $|c(X_{ij}^s)| = 1$ holds for each $j \in [n_i]$.

Lemma 2.1 Let $c = \{C_1,\ldots,C_k\}$ be a strict k-coloring of $H_{n_1,\ldots,n_s}$ with $k \geq t$. For each $i \in [s]$, if $|c(X_{ij}^s)| = 1$ for some $j \in [n_i]$, then for each $h \in [n_i]$, $|c(X_{ih}^s)| = 1$.

Proof Without loss of generality, suppose $X_{11}^s \subseteq C_1$.

We claim that for each $j \in [n_1] \setminus \{1\}$, there exists an integer in $[k] \setminus \{1\}$, say l_j, such that $X_{1j}^s \subseteq C_1 \cup C_{l_j}$. Suppose for a contradiction that $X_{1j}^s \cap C_p \neq \varnothing$ and $X_{1j}^s \cap C_q \neq \varnothing$ for some $j \in [n_1] \setminus \{1\}$ and $p, q \in [k] \setminus \{1\}$ with $p \neq q$. Without loss of generality, suppose $p = 2, q = 3$. Pick $(j, i_2, \ldots, i_s) \in C_2$, $(j, j_2, \ldots, j_s) \in C_3$ and $\alpha_h \in C_h$ for each $h \in \{4,\ldots,t\}$. Note that $(1, i_2, \ldots, i_s) \in C_1$. Then

$$\{(1,i_2,\ldots,i_s),(j,i_2,\ldots,i_s),(j,j_2,\ldots,j_s),\alpha_4,\ldots,\alpha_t\}$$

is an edge and is polychromatic, a contradiction. Hence, our claim is valid.

Suppose $X_{1a}^s \subseteq C_1 \cup C_p$ and $X_{1a}^s \cap C_1 \neq \varnothing, X_{1a}^s \cap C_p \neq \varnothing$ for some $a \in [n_1] \setminus \{1\}$ and $p \in [k] \setminus \{1\}$. Without loss of generality, suppose $p = 2$ and pick $(a, i_2, \ldots, i_s) \in C_1$, $(a, j_2, \ldots, j_s) \in C_2$. Note that $k \geq t$. Assume that $X_{1q}^s \subseteq C_1 \cup C_3$ and $X_{1q}^s \cap C_3 \neq \varnothing$, where $q \in [n_1] \setminus \{1, a\}$. Pick $\alpha_h \in C_h, h = 4,\ldots,t$. If $(q, i_2, \ldots, i_s) \in C_3$, then the edge

$$\{(a,i_2,\ldots,i_s),(a,j_2,\ldots,j_s),(q,i_2,\ldots,i_s),\alpha_4,\ldots,\alpha_t\}$$

is polychromatic, a contradiction; or the edge

$$\{(a,i_2,\ldots,i_s),(a,j_2,\ldots,j_s),(q,j_2,\ldots,j_s),\alpha_4,\ldots,\alpha_t\}$$

is polychromatic if $(q, j_2, \ldots, j_s) \in C_3$, also a contradiction. Hence, $(q,i_2,\ldots,i_s),(q,j_2,\ldots,j_s) \in C_1$. Pick $(q,k_2,\ldots,k_s) \in C_3$. Note that $\beta = (a,k_2,\ldots,k_s) \in C_1 \cup C_2$. Then the edge

$$\{(a,i_2,\ldots,i_s),(q,k_2,\ldots,k_s),\beta,\alpha_4,\ldots,\alpha_t\}$$

is polychromatic if $\beta \in C_2$; or the edge

$$\{(a,j_2,\ldots,j_s),(q,k_2,\ldots,k_s),\beta,\alpha_4,\ldots,\alpha_t\}$$

is polychromatic if $\beta \in C_1$, a contradiction. Therefore, for each $h \in [n_1]$, $|C(X_{ih}^s)| = 1$, as desired.

In the following, we shall prove that under any strict coloring of $H_{n_1,\ldots,n_s}$ with at least t colors, there exists a X_{ij}^s which is contained in one color class.

Lemma2.2 Let $c = \{C_1, C_2, \ldots, C_k\}$ be a strict k-coloring of $H_{n_1,\ldots,n_s}$ with $k \geq t$. Then there exist integers $i \in [s]$ and $j \in [n_i]$ such that $|c(X_{ij}^s)| = 1$.

Proof We prove it by induction on s. Let $s = 2$. Suppose for a contradiction that there exist $i \in [n_1]$ and $j \in [n_2]$ such that X_{1i}^2 Ú C_l and X_{2j}^2 Ú C_l for every $l \in [k]$. Without loss of generality, suppose $i = j = 1$, $(1,1) \in C_1, (1,2) \in C_2$, $(a,1) \notin C_1$. Pick $\alpha_h \in C_h, 3 \leq h \leq t-1$. Since $\{(1,1),(1,2),(a,1),\alpha_3,\ldots,\alpha_{t-1}\}$ is an edge, $(a,1) \in C_2 \cup \cdots \cup C_{t-1}$. Pick $(p,q) \in C_t$. Assume that $(a,1) \in C_2$. Then since $\{(1,1),(p,q),(1,q),\alpha_3,\ldots,\alpha_{t-1}\}$ and $\{(1,1),(a,1),(1,q),\alpha_3,\ldots,\alpha_{t-1}\}$ are edges, one gets that $(1,q) \in C_h$ for some $h \in [t-1] \setminus \{2\}$. If $h = 1$, then the edge $\{(1,q),(1,2),(p,q),\alpha_3,\ldots,\alpha_{t-1}\}$ is polychromatic; if $h \neq 1$, then the edge $\{(1,1),(1,2),(p,q),\alpha_3,\ldots,\alpha_{h-1},(1,q),\alpha_{h+1},\ldots,\alpha_{t-1}\}$ is polychromatic, a contradiction. Assume that $(a,1) \in C_h$ for some $h \in \{3,\ldots,t-1\}$. Then the edge $\{(1,1),(1,2),\alpha_3,\ldots,\alpha_{h-1},(a,1),\alpha_{h+1},\ldots,\alpha_{t-1},(p,q)\}$ is polychromatic, a contradiction. It follows that the conclusion is true for the case of $s=2$. Assume that the conclusion is also true for the case of $s-1$.

Let $X'=\{(x_2,x_2,x_3,x_4,\ldots,x_s) \mid x_j\in[n_j], j\in[s]\setminus\{1\}\}$. Then

ϕ: $X'\to X_{n_2,\ldots,n_s}$, $(x_2,x_2,x_3,\ldots,x_s)\mapsto(x_2,\ldots,x_s)$

is an isomorphism from $H'=H_{n_1,\ldots,n_s}[X']$ to $H_{n_2,\ldots,n_s}$.

Note that the restriction of any strict coloring of $H_{n_1,\ldots,n_s}$ on X' corresponds to a strict k_1-coloring of $H_{n_2,\ldots,n_s}$. We focus on the colors of the vertices of X' and get the following three possible cases.

Case 1 $k_1=1$.

That is to say $X'\subseteq C_h$ for some $h\in[k]$.Without loss of generality, suppose $X'\subseteq C_1$. Pick $\alpha_i=(a_{i1},\ldots,a_{is})\in C_i$ for each $i\in[t]\setminus\{1\}$. If there exist two distinct integers in $[t]\setminus\{1\}$, say p, q, such that $a_{p1}=a_{q1}$, then $\{(a_{22},a_{22},a_{23},\ldots,a_{2s}),\alpha_i \mid i\in[t]\setminus\{1\}\}$ is an edge and is polychromatic, a contradiction. Hence, $a_{p1}\neq a_{q1}$ if $p\neq q$. For each $j\in[t]\setminus\{1,2\}$, from the edge $\{(a_{21},a_{32},\ldots,a_{3s}),(a_{22},a_{22},a_{23},\ldots,a_{2s}),\alpha_i \mid i\in[t]\setminus\{1,j\}\}$, we have $(a_{21},a_{32},\ldots,a_{3s})\notin C_j$. Then from the edge $\{(a_{21},a_{32},\ldots,a_{3s}),\alpha_i \mid i\in[t]\setminus\{1\}\}$, we have $(a_{21},a_{32},\ldots,a_{3s})\in C_2$. For each $\alpha=(a_{21},x_2,\ldots,x_s)\in X^s_{1a_{21}}$, from the edge $\{\alpha,(a_{21},a_{32},\ldots,a_{3s}),(a_{32},a_{32},a_{33},\ldots,a_{3s}),\alpha_i \mid i\in[t]\setminus\{1,2,j\}\}$, we have $\alpha\notin C_j$ for all $j\in[t]\setminus\{1,2\}$, then $\alpha\in C_2$ since $\{\alpha,(a_{21},a_{32},\ldots,a_{3s}),\alpha_i \mid i\in[t]\setminus\{1,2\}\}$ is an edge. Therefore, $X^s_{1a_{21}}\subseteq C_2$, as desired.

Case 2 $2\le k_1\le t-1$.

Without loss of generality, suppose $X'\subseteq C_1\cup\cdots\cup C_{k_1}$, $X'\cap C_i\neq\varnothing$ for all $i\in[k_1]$. Pick $\alpha_i=(a_{i2},a_{i2},a_{i3},\ldots,a_{is})\in C_i$, $i=1,\ldots,k_1$, and $\beta_j=(b_{j1},\ldots,b_{js})\in C_j$, $j=k_1+1,\ldots,t$. Then $(b_{j2},b_{j2},b_{j3},\ldots,b_{js})\in C_1\cup\cdots\cup C_{k_1}$ for all $j\in[t]\setminus[k_1]$.

Case 2.1 $(b_{t2},b_{t2},b_{t3},\ldots,b_{ts})\in C_1$. If $a_{p2}=b_{q1}$ for some $p\in[k_1]\setminus\{1\}$ and $q\in[t]\setminus[k_1]$, then $\{(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1\}, j\in[t]\setminus[k_1]\}$ is an edge and is polychromatic, a contradiction. Hence,

$a_{i2}\neq b_{j1}$ for all $i\in[k_1]\setminus\{1\}$ and $j\in[t]\setminus[k_1]$.

For each $p\in[k_1]\setminus\{2\}$, since $\{(a_{22},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{p\}, j\in[t]\setminus[k_1]\}$ is an edge, $(a_{22},b_{t2},\ldots,b_{ts})\notin C_p$; and for each $q\in[t-1]\setminus[k_1]$, from the edge $\{(a_{22},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1], j\in\{k_1+1,\ldots,t\}\setminus\{q\}\}$, we get that $(a_{22},b_{t2},\ldots,b_{ts})\notin C_q$. Then the edges $\{(a_{22},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1\}, j\in[t-1]\setminus[k_1]\}$ and $\{(a_{22},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1\}, j\in[t]\setminus[k_1]\}$ imply that $(a_{22},b_{t2},\ldots,b_{ts})\in C_2$. Pick $\eta=(a_{22},x_2,\ldots,x_s)\in X^s_{1a_{22}}$. Since $\{\eta,(a_{22},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,2,p\}, j\in[t]\setminus[k_1]\}$ is an edge, $(a_{22},b_{t2},\ldots,b_{ts})\notin C_p$ for all $p\in[k_1]\setminus\{1,2\}$; since $\{\eta,(a_{22},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,2\}, j\in\{k_1+1,\ldots,t\}\setminus\{q\}\}$ is an edge, $\eta\notin C_q$ for each $q\in[t]\setminus[k_1]$. Then the edge $\{\eta,(a_{22},a_{t2},\ldots,a_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,2\}, j\in[t]\setminus[k_1]\}$ implies that $\eta\in C_2$. Therefore, $X^s_{1a_{22}}\subseteq C_2$, as desired.

Case 2.2 $(b_{t2},b_{t2},b_{t3},\ldots,b_{ts})\in C_h$ for some $h\in[k_1]\setminus\{1\}$. If $a_{p2}=b_{q1}$ for some $p\in[k_1]\setminus\{h\}$ and $q\in[t]\setminus[k_1]$, then $\{(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{h\}, j\in[t]\setminus[k_1]\}$ is an edge and is polychromatic. Hence, $a_{i2}\neq b_{j1}$ for all $i\in[k_1]\setminus\{h\}$ and $j\in[t]\setminus[k_1]$.

For each $p\in[k_1]\setminus\{1,h\}$, since $\{(a_{12},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{p\}, j\in[t]\setminus[k_1]\}$ is an edge, $(a_{12},b_{t2},\ldots,b_{ts})\notin C_p$; for each $q\in\{k_1+1,\ldots,t-1\}$, from the edge $\{(a_{12},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1], j\in\{k_1+1,\ldots,t\}\setminus\{q\}\}$, we get $(a_{12},b_{t2},\ldots,b_{ts})\notin C_q$. Then the edges $\{(a_{12},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{h\}, j\in[t-1]\setminus[k_1]\}$ and $\{(a_{12},b_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{h\}, j\in[t]\setminus[k_1]\}$ imply that $(a_{12},b_{t2},\ldots,b_{ts})\in C_1$. Pick $\eta=(a_{12},x_2,\ldots,x_s)\in X^s_{1a_{12}}$. Since $\{\eta,(a_{12},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,h,p\}, j\in[t]\setminus[k_1]\}$ is an edge, $(a_{12},a_{t2},\ldots,a_{ts})\notin C_p$ for each $p\in[k_1]\setminus\{1,h\}$; and from the edge $\{\eta,(a_{12},b_{t2},\ldots,b_{ts}),(b_{t2},b_{t2},b_{t3},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,h\}, j\in[t]\setminus[k_1], j\neq q\}$, we get that $\eta\notin C_q$ for each $q\in[t]\setminus[k_1]$. Then the edge $\{\eta,(a_{12},bb_{t2},\ldots,b_{ts}),\alpha_i,\beta_j \mid i\in[k_1]\setminus\{1,h\}, j\in[t]\setminus[k_1]\}$ implies that $\eta\in C_1$. Therefore, $X^s_{1a_{12}}\subseteq C_1$, as desired.

Case 3 $t\le k_1\le k$.

By induction, it is true for $H[X']$. That is to say, there exists an $i\in[s]\setminus\{1\}$ such that for each $j\in[n_i]$, $X_{ij'}\subseteq C_{l_j}$ for some $l_j\in[k_1]$, where $X_{ij'}=X'\cap X^s_{ij}$.

Case 3.1 $i=2$.

Without loss of generality, suppose $X_{2j'} \subseteq C_j, j = 1, \ldots, t$. For each $j \in [t] \backslash \{1\}$ and $(1, j, x_3, \ldots, x_s) \in X_{n_1,\ldots,n_s}$, since $\{(1, j, x_3, \ldots, x_s), (i, i, x_3, \ldots, x_s) | \ i \in [t] \backslash \{p\}\}, p \in [t] \backslash \{1, j\}$ are edges, one has $(1, j, x_3, \ldots, x_s) \in C_1 \cup C_j$. Similarly, we have $(i, j, x_3, \ldots, x_s) \in C_i \cup C_j$ for any $i, j \in [t]$ and $(i, j, x_3, \ldots, x_s) \in X_{n_1,\ldots,n_s}$.

Assume that $(1, 2, 1, \ldots, 1) \in C_1$. From the edge $\{(1, 2, 1, \ldots, 1), (1, 3, 1, \ldots, 1), (i, i, 1, \ldots, 1) | \ i \in [t] \backslash \{1, 3\}\}$, we get $(1, 3, 1, \ldots, 1) \in C_1$. Pick $\eta = (1, x_2, \ldots, x_s) \in X_{11}^s$. From the edges $\{\eta, (1, 2, 1, \ldots, 1), (2, 2, 1, \ldots, 1), (i, i, 1, \ldots, 1) | \ i \in [t] \backslash \{1, 2, p\}\}, p \in [t] \backslash \{1, 2\}$, $\{\eta, (1, 3, 1, \ldots, 1), (3, 3, 1, \ldots, 1), (i, i, 1, \ldots, 1) | \ i \in [t] \backslash \{1, 3, q\}\}, q \in [t] \backslash \{1, 3\}$, we have $\eta \in C_1$, which implies that $X_{11}^s \subseteq C_1$.

Assume that $(1, 2, 1, \ldots, 1) \in C_2$. Then we may similarly get that $X_{22}^s \subseteq C_2$, as desired.

Case 3.2 $i \in [s] \backslash \{1, 2\}$.

Suppose $i = 3$ and $X_{3i'} \subseteq C_i, i = 1, \ldots, t$. Pick $\eta = (x_1, x_2, 1, x_4, \ldots, x_s) \in X_{31}^s$. Then $\eta \notin C_j$ for each $j \in [t] \backslash \{1\}$, since $\{\eta, (x_2, x_2, i, x_4, \ldots, x_s) | \ i \in [t] \backslash \{j\}\}$ is an edge. From the edge $\{\eta, (x_1, x_2, i, x_4, \ldots, x_s) | \ i \in [t-1]\}$, we have $\eta \in C_1$, which implies that $X_{31}^s \subseteq C_1$.

Combining Lemmas 2.1 and 2.2, we get the following result.

Theorem 2.3 Let $s \geq 2$ and $n_1 \geq \cdots \geq n_s \geq t$ be integers. Then

(1) $\bar{\chi}(H_{n_1,\ldots,n_s}) = n_1$;

(2) $r_k = \sum_{i=1}^{s} S(n_i, k)$ for $t \leq k \leq n_1$, where r_k is the k-th entry of the chromatic spectrum of $H_{n_1,\ldots,n_s}$.

In the following, we give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Let $(q_1, \ldots, q_l) = (\underbrace{n_1, \ldots, n_1}_{m_1}, \ldots, \underbrace{n_s, \ldots, n_s}_{m_s})$ and $H = H_{q_1,\ldots,q_l}$. From the above discussion, we get that $\bar{\chi}(H) = n_1$ and $r_k = m_1 S(n_1, k) + m_2 S(n_2, k) + \cdots + m_s S(n_s, k), k = t, \ldots, n_1$, where r_k is the k-th entry of the chromatic spectrum of H. It follows that H is a desired C-hypergraph. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 For integer n at least three, let $X = X_{n,t}$, $C' = \{\{(1,1), (1,2), \ldots, (1,t)\}\}, C = C_{n,t} \cup C'$ and $H = (X, C)$. Note that $H_{n,t}$ is a spanning sub-hypergraph of H and c_1^2 is a strict n-coloring of H. But the edge $\{(1,1), \ldots, (1,t)\}$ is polychromatic under c_2^2. Hence, c_2^2 is not a strict coloring of H. Therefore, for any $k \geq t$, H has $S(n,k)$ strict k-colorings getting from c_1^2. It follows that H is a t-uniform C-hypergraph with upper chromatic number n and $r_k = S(n,k)$ for any $t \leq k \leq n$, as desired.

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