On the Chromatic Spectrum of Uniform C-Hypergraphs

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Abstract—Motivated by solving the problem proposed by Král, (Electronic J. Combin. 11#R19, 2004), we construct a family of uniform C-hypergraphs whose chromatic spectrum is a linear combination of the Stirling numbers of the second kind.

Keywords- C-hypergraph; chromatic spectrum; Stirling number of the second kind

I. INTRODUCTION

A mixed hypergraph on a finite set \( X \) is a triple \( H=(X,C,D) \), where \( C \) and \( D \) are families of subsets of \( X \). The members of \( C \) and \( D \) are called \( C \)-edges and \( D \)-edges, respectively. A mixed hypergraph with \( D=\emptyset \), denoted by \( H=(X,C) \), is called a \( C \)-hypergraph, and a hypergraph with \( C=\emptyset \), denoted by \( H=(X,D) \), is called a \( D \)-hypergraph. A mixed hypergraph is \( t \)-uniform if every edge contains \( t \) vertices.

A proper \( k \)-coloring of a mixed hypergraph \( H \) is a mapping from \( X \) into a set of \( k \) colors such that each \( C \)-edge has two vertices with a Common color and each \( D \)-edge has two vertices with \( \emptyset \) colors. A coloring may also be viewed as a partition (feasible partition) of \( X \), where the color classes (partition classes) are the sets of vertices assigned to the same color. A strict \( k \)-coloring is a proper \( k \)-coloring with \( k \) nonempty color classes, and a mixed hypergraph is \( k \)-colorable if it has a strict \( k \)-coloring. The maximum (resp. minimum) number of the colors in a strict coloring of \( H=(X,C,D) \) is the upper chromatic number (resp. lower chromatic number) of \( H \), denoted by \( \chi^ U(H) \) (resp. \( \chi^ L(H) \)).

The set of all the values \( k \) such that \( H \) has a strict \( k \)-coloring is called the feasible set of \( H \), denoted by \( F(H) \). For each \( k \), let \( r_k \) denote the number of feasible partitions of the vertex set. The vector \( R(H)=(r_1,r_2,\ldots,r_t) \) is called the chromatic spectrum of \( H \), where \( t \) is the upper chromatic number of \( H \). The study of the colorings of mixed hypergraphs has made a lot of progress since its inception [5]. For more information, see [4,6].

It is readily seen that \( H \) is a \( C \)-hypergraph if and only if \( 1 \in F(H) \). Moreover, if \( r_i \neq 0 \), then \( r_i = 1 \). For the case \( 1 \notin F(H) \), Jiang et al. [3] proved that for any finite set \( S \) of integers greater than 1, there exists a mixed hypergraph \( H \) such that \( F(H)=S \). Král [2] strengthened this result by showing that prescribing any positive integer \( r_i \), there exists a mixed hypergraph which has precisely \( r_i \) strict \( k \)-colorings for all \( k \in S \).

On the chromatic spectrum of \( C \)-hypergraphs

Let \( S(n,k) \) be the Stirling number of the second kind, i.e., the number of ways to partition a set of \( n \) elements into \( k \) nonempty subsets. Assume that \( H=(X,C) \) be a \( t \)-uniform \( C \)-hypergraph and \( e=(C_1,C_2,\ldots,C_t) \) a strict \( n \)-coloring of \( H \). Then for any \( t \leq k \leq n \), we can get \( S(n,k) \) strict \( k \)-colorings of \( H \) by union some color classes of \( e \), i.e., \( r_k \geq S(n,k) \). In this sense, the \( n \)-dimensional vector with the \( k \)-th entry being \( S(n,k) \) for \( t \leq k \leq n \) is a lower bound on the chromatic spectrum of \( C \)-hypergraphs with upper chromatic number \( n \). Zhang et al. [7] proved that this lower bound can be attained for 3-uniform \( C \)-hypergraphs. In this paper, we generalize it to \( t \)-uniform \( C \)-hypergraphs and get the following results.

Theorem 1.1 Let \( n_1 > \cdots > n_t \geq t \geq 3 \) be integers. Then for any positive integers \( m_1,\ldots,m_t \) and \( t_1 := m_0S(n_1,k)+m_1S(n_2,k)+\cdots+m_tS(n_t,k) \), \( k=t,\ldots,n_1 \), there exists a \( t \)-uniform \( C \)-hypergraph \( H \) with upper chromatic number \( n_1 \) and \( r_k = t_k \) for each \( k \in \{t,\ldots,n_1\} \), where \( r_k \) is the \( k \)-th entry of the chromatic spectrum of \( H \).
Theorem 1.2 For any integers $n \geq t \geq 3$, there exists a $t$-uniform $C$-hypergraph $H$ satisfies that $\chi (H) = n$ and $R(H) = (r_1, r_2, \ldots, r_s, S_1(n, t), \ldots, S_t(n, n))$.

II. PROOF OF THE THEOREMS

For any positive integer $n$, let $[n]$ denote the set $\{1, \ldots, n\}$.

We first give the basic construction of $t$-uniform $C$-hypergraphs. For integers $s \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_t \geq t$, let

\[
X_{n_1, \ldots, n_t} = \{(x_{11}, x_{12}, \ldots, x_{1s}) \mid x_{1i} \in [n_i], i \in [s] \},
\]
\[
C_{n_1, \ldots, n_t} = \{[\alpha_1, \ldots, \alpha_t] \mid \alpha_i \in X_{n_1, \ldots, n_t}, i \in [t] \}
\]

where $[\alpha_1, \ldots, \alpha_t] = (\alpha_{i1}, \ldots, \alpha_{it})$, and $\alpha_{ij}$ is the $j$-th component of $\alpha_i$.

Moreover, for any $k \geq t$, we can get $S(n, k)$ strict $k$-colorings of $H_{n_1, \ldots, n_t}$ from $C_{n_1, \ldots, n_t}$. Hence, $H_{n_1, \ldots, n_t}$ has at least $S(n, k) + \cdots + S(n, k)$ strict $k$-colorings. In the following, we shall prove that $H_{n_1, \ldots, n_t}$ has no other strict $k$-colorings, that is to say, $r_k = S(n, k) + \cdots + S(n, k)$ for every $k \leq n_1$.

Let $k \in \{1, \ldots, n_1\}$, $i \in [s]$ and $c = \{C_1, \ldots, C_k\}$ be one of the $S(n, k)$ strict $k$-colorings of $H_{n_1, \ldots, n_t}$ getting from $C_{n_1, \ldots, n_t}$. Note that for each $j \in [n_i]$, there exists an integer in $[k]$, say $l_{ij}$, such that $X_{ij} \subseteq C_{l_{ij}}$. It suffices to prove that for any strict $k$-coloring $c = \{C_1, \ldots, C_k\}$ of $H_{n_1, \ldots, n_t}$, there exists an $i \in [s]$ such that $|c(X_{ij})| = 1$ holds for each $j \in [n_i]$.

Lemma 2.1 Let $c = \{C_1, \ldots, C_k\}$ be a strict $k$-coloring of $H_{n_1, \ldots, n_t}$ with $k \geq t$. For each $i \in [s]$, if $|c(X_{ij})| = 1$ for some $j \in [n_i]$, then for each $h \in [n_i]$, $|c(X_{ih})| = 1$.

Proof Without loss of generality, suppose $X_{i1} \subseteq C_1$.

We claim that for each $j \in [n_i] \setminus \{i\}$, there exists an integer in $[k] \setminus \{i\}$, say $l_j$, such that $X_{ij} \subseteq C_{l_j} \cup C_i$. Suppose for a contradiction that $X_{ij} \cap C_i \neq \emptyset$ and $X_{ij} \cap C_j \neq \emptyset$ for some $j \in [n_i] \setminus \{i\}$ and $p, q \in [k] \setminus \{i\}$ with $p \neq q$. Without loss of generality, suppose $p = 2, q = 3$. Pick $(j_1, j_2, \ldots, j_t) \in C_i$ and $\alpha_i \in C_i$ for each $h \in \{4, \ldots, t\}$. Note that $(1, i_2, \ldots, i_t) \in C_i$. Then $(1, i_2, \ldots, i_t), (j_1, i_2, \ldots, i_t), (j, j_1, \ldots, j_t), \alpha_i, \ldots, \alpha_t$ is an edge and is polychromatic, a contradiction. Hence, our claim is valid.

Suppose $X_{i1} \subseteq C_1 \cup C_p$ and $X_{i1} \cap C_j \neq \emptyset, X_{i1} \cap C_p \neq \emptyset$ for some $a \in [n_i \setminus \{i\}] \setminus \{j\}$ and $p \in [k] \setminus \{i\}$. Without loss of generality, suppose $p = 2$ and pick $(a,i_2, \ldots, i_t) \in C_1 \cup (a, j_2, \ldots, j_t) \in C_2$. Note that $k \geq t$. Assume that $X_{i1} \subseteq C_1 \cup C_j$ and $X_{i1} \cap C_p \neq \emptyset$ where $q \in [n_i \setminus \{a, j\}]$. Pick $\alpha_j \in C_j, h \in \{4, \ldots, t\}$. If $(q, j_2, \ldots, j_t) \in C_j$, then the edge $(q, i_2, \ldots, i_t), (a, j_2, \ldots, j_t), (j_1, j_2, \ldots, j_t), \alpha_j, \ldots, \alpha_t$ is polychromatic, a contradiction; or the edge $(q, i_2, \ldots, i_t), (a, j_2, \ldots, j_t), (q, j_1, \ldots, j_t), \alpha_j, \ldots, \alpha_t$ is polychromatic if $(q, j_1, \ldots, j_t) \in C_j$, also a contradiction. Hence, $(a, i_2, \ldots, i_t), (q, j_1, \ldots, j_t), (q, j_2, \ldots, j_t), \alpha_j, \ldots, \alpha_t$ is polychromatic if $p \in C_j$; or the edge $(a, i_2, \ldots, i_t), (q, j_1, \ldots, j_t), (q, k_2, \ldots, k_t), \beta, \alpha_j, \ldots, \alpha_t$ is polychromatic if $p \notin C_j$, a contradiction. Therefore, for each $h \in [n_i \setminus \{a, j\}], |c(X_{ih})| = 1$, as desired.

In the following, we shall prove that under any strict coloring of $H_{n_1, \ldots, n_t}$ with at least $t$ colors, there exists a $X_{ij}$ which is contained in one color class.

Lemma 2.2 Let $c = \{C_1, C_2, \ldots, C_t\}$ be a strict $t$-coloring of $H_{n_1, \ldots, n_t}$ with $k \geq t$. Then there exist integers $i \in [s]$ and $j \in [n_i]$ such that $|c(X_{ij})| = 1$.

Proof We prove it by induction on $s$. Let $s = 2$. Suppose for a contradiction that there exist $i \in [n_1]$ and $j \in [n_i]$ such that $X_{ij} \subseteq C_i$ and $X_{ij} \subseteq C_j$ for every $i \in [k]$. Without loss of generality, suppose $i = j = 1$. $(1, l_1) \in C_1, (1, l_2) \in C_2, (a_1) \notin C_1$. Pick $\alpha_{a_1} \in C_1, 3 \leq h \leq t - 1$. Since $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1})$ is an edge, $(a_1) \in C_1 \cup \cdots \cup C_{a_1}$. Pick $(p, q) \in C_1$. Assume that $(a_1) \in C_1$. Then since $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1}) \notin C_1$ and $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1})$ are edges, one gets that $(1, q) \in C_1$ for some $h \in [t - 1] \setminus \{2\}$. If $h = 1$, then the edge $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1})$ is polychromatic; if $h \neq 1$, then the edge $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1})$ is polychromatic, a contradiction. Assume that $(a_1) \in C_1$ for some $h \in \{3, \ldots, t - 1\}$. Then the edge $(\alpha_1, (1, l_2), (a_1), \alpha_{a_1}, \ldots, \alpha_{a_1}, (p, q))$ is polychromatic, a contradiction. It follows that the conclusion holds.
is true for the case of \( s = 2 \). Assume that the conclusion is also true for the case of \( s - 1 \).

Let \( X' = (x_1, x_2, x_3, \ldots, x_n) \mid \forall x_i \in [n], x_j \in [s]\{1]\) \}. Then \( \phi: X' \rightarrow X_{n-s-n}, (x_1, x_2, x_3, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n) \) is an isomorphism from \( H' = H_{n-s-n}[X'] \) to \( H_{n-s-n} \).

Note that the restriction of any strict coloring of \( H_{n-s-n} \) on \( X' \) corresponds to a strict \( k_i \)-coloring of \( H_{n-s-n} \). We focus on the colors of the vertices of \( X' \) and get the following three possible cases.

**Case 1** \( k_i = 1 \).

That is to say \( X' \subseteq C_i \) for some \( h \in [k_i] \). Without loss of generality, suppose \( X' \subseteq C_i \). Pick \( a_1 = (a_1, a_2, \ldots, a_n) \in C_i \) for each \( i \in [r]\{1\} \). If there exist two distinct integers in \( [r]\{1\} \), say \( p, q \), such that \( a_{pq} = a_{pq} \), then

\[
\{(a_{12}, a_{22}, a_{32}, \ldots, a_{n2}, a_1) \mid i \in [r]\{1, j\}\}
\]

is an edge and is polychromatic, a contradiction. Hence, \( a_{pq} \neq a_{pq} \) if \( p \neq q \). For each \( j \in [r]\{1, 2\} \) from the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid j \in [1, j]\},
\]

we have \( (a_{12}, a_{22}, \ldots, a_{n2}) \in C_j \). Then from the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [r]\{1\}\}
\]

we have \( (a_{12}, a_{22}, \ldots, a_{n2}) \in C_2 \). For each \( \alpha = (a_1, x_2, \ldots, x_n) \in X_{1+i} \), from the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid j \in [1, 2, j]\},
\]

we have \( \alpha \in C_j \) for all \( j \in [r]\{1, 2\} \). Then \( \alpha \in C_2 \) since

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [r]\{1, 2\}\}
\]

is an edge. Therefore, \( X_{1+i} \subseteq C_2 \) as desired.

**Case 2** \( 2 \leq k_i \leq t - 1 \).

Without loss of generality, suppose \( X' \subseteq C_1 \cup \ldots \cup C_t \). \( X' \cap C_i \neq \emptyset \) for all \( i \in [k_i] \). Pick \( a_i = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}) \in C_i \) for \( i = 1, \ldots, k_i \) and \( \beta_i = (b_{i1}, \ldots, b_{in}) \in C_i \), \( j = k_i + 1, \ldots, t \). Then

\[
(b_2, b_2, b_3, \ldots, b_n) \in C_1 \cup \ldots \cup C_t
\]

for all \( j \in [r]\{k_i\} \).

**Case 2.1** \( (b_2, b_2, b_3, \ldots, b_n) \in C_1 \). If \( a_{pq} = b_{pq} \) for some \( p \in [k_i]\{1\} \) and \( q \in [r]\{k_i\} \), then

\[
\{(b_1, b_2, b_2, b_3, \ldots, b_n) \mid i \in [k_i]\{1\}, j \in [r]\{k_i\}\}
\]

is an edge and is polychromatic, a contradiction. Hence, \( a_{pq} \neq b_{pq} \) for all \( i \in [k_i]\{1\} \) and \( j \in [r]\{k_i\} \).

For each \( p \in [k_i]\{2\} \), since

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{p\}, j \in [r]\{k_i\}\}
\]

is an edge, \( (a_{12}, a_{22}, \ldots, a_{n2}) \notin C_p \); and for each \( q \in [r\{1\}\{k_i\} \), from the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

we get that \( (a_{12}, a_{22}, \ldots, a_{n2}) \notin C_q \). Then the edges

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

and

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

imply that \( (a_{12}, a_{22}, \ldots, a_{n2}) \in C_i \). Pick \( \eta = (a_{12}, \ldots, x_n) \in X_{1+i} \). Since

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

is an edge, \( (a_{12}, a_{22}, \ldots, a_{n2}) \in C_p \) for all \( p \in [k_i]\{1\} \); since

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

is an edge, \( \eta \notin C_q \) for each \( q \in [r]\{k_i\} \). Then the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1, l\}, j \in [r]\{k_i\} \}
\]

implies that \( \eta \notin C_i \). Therefore, \( X_{1+i} \subseteq C_2 \) as desired.

**Case 2.2** \( (b_2, b_2, b_3, \ldots, b_n) \in C_1 \) for some \( h \in [k_i]\{1\} \). If \( a_{pq} = b_{pq} \) for some \( p \in [k_i]\{h\} \) and \( q \in [r]\{k_i\} \), then

\[
\{(b_1, b_2, b_3, \ldots, b_n) \mid i \in [k_i]\{h\}, j \in [r]\{k_i\} \}
\]

is an edge and is polychromatic. Hence, \( a_{pq} \neq b_{pq} \) for all \( i \in [k_i]\{h\} \) and \( j \in [r]\{k_i\} \).

For each \( p \in [k_i]\{1, h\} \), since

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{p\}, j \in [r]\{k_i\} \}
\]

is an edge, \( (a_{12}, a_{22}, \ldots, a_{n2}) \notin C_p \) for each \( p \in [k_i]\{1, h\} \); and from the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

we get that \( (a_{12}, a_{22}, \ldots, a_{n2}) \notin C_q \). Then the edges

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

and

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1\}, j \in [r\{1\}\{k_i\} \}
\]

imply that \( \eta \notin C_q \) for each \( q \in [r]\{k_i\} \). Then the edge

\[
\{(a_{12}, a_{22}, \ldots, a_{n2}) \mid i \in [k_i]\{1, h\}, j \in [r]\{k_i\} \}
\]

implies that \( \eta \notin C_i \). Therefore, \( X_{1+i} \subseteq C_2 \), as desired.

**Case 3** \( t \leq k_i \leq k \).

By induction, it is true for \( H'[X'] \). That is to say, there exists an \( j \in [s]\{1\} \) such that for each \( j \in [n] \), \( X_{ij} \subseteq C_j \) for some \( l_j \in [k_i] \), where \( X' = X' \cap X_j \).

**Case 3.1** \( i = 2 \).
Without loss of generality, suppose \( X_{x_j} \subseteq C_j, j = 1, \ldots, t \).

For each \( j \in [r]\setminus\{1\} \) and \( (i, j, x_1, \ldots, x_r) \in X_{n_i} \), since \( \{(i, j, x_1, \ldots, x_r) \mid i \in [r]\setminus\{p\}\}, p \in [r]\setminus\{1, j\} \) are edges, one has \( (i, j, x_1, \ldots, x_r) \in C_i \cup C_j \). Similarly, we have \((i, j, x_1, \ldots, x_r) \in C_i \cup C_j \) for any \( i, j \in [r] \) and \((i, j, x_1, \ldots, x_r) \in X_{n_i} \), as desired.

Therefore, \( -C \mid \iota \subseteq \{1, 2\} \). Then we may similarly get \( \iota \subseteq C_i \), as desired.

**Case 3.2** \( i \in [s]\setminus\{1, 2\} \).

Suppose \( i \geq 3 \) and \( X_{x_i} \subseteq C_i, i = 1, \ldots, t \). Pick \( \eta = (x_i, x_1, x_2, x_3, \ldots, x_r) \in X_{x_i} \). Then \( \eta \not\in C_i \) for each \( j \in [r]\setminus\{1\} \), since \( \{\eta, (x_i, x_1, x_2, x_3, \ldots, x_r) \mid i \in [r]\setminus\{j\}\} \) is an edge. From the edge \( \{\eta, (x_i, x_1, x_2, x_3, \ldots, x_r) \mid i \in [t-1]\} \), we have \( \eta \in C_i \), which implies that \( X_{x_i} \subseteq C_i \).

Assume that \((1, 2, 1, \ldots, 1) \in C_i \). Then we may similarly get

\[ X_{x_i} \subseteq C_i \], as desired.

**Proof of Theorem 1.2** For integer \( n \) at least three, let \( X = X_n, C = \{(1,1), (1,2), \ldots, (1,t)\} \) be \( C \cup C \). Note that \( H_{n/} \) is a spanning sub-hypergraph of \( H \) and \( \chi_i \) is a strict \( n \)-coloring of \( H \). But the edge \( \{(1,1), \ldots, (1,t)\} \) is isomorphic under \( \chi_i \). Hence, \( \chi_i \) is not a strict coloring of \( H \). Therefore, for any \( k \geq t \), \( H \) has \( S(n,k) \) strict \( k \)-colorings getting from \( \chi_i \). It follows that \( H \) is a \( 1 \)-uniform \( C \)-hypergraph with upper chromatic number \( n \) and \( r_k = S(n,k) \) for any \( t \leq k \leq n \), as desired.

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