A New Trigonometrically Fitted Modified TDRK Methods for Oscillatory Differential Equations

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Abstract—**The classical Runge-Kutta (RK) methods are very popular in solving oscillatory differential equations due to their simple iterative form and high efficiency. In the recent years, a new type RK method, two-derivative Runge-Kutta (TDRK) method was proposed, which can reach higher order with few function evaluations. This paper aims at developing, analyzing, and validating a new trigonometrically fitted modified TDRK method for oscillatory differential equations. Furthermore, phase properties of the new method is examined. Numerical comparisons with some state-of-the-art high efficient solvers are reported.**

***Keywords-** Two-derivative Runge-Kutta method; trigonometrically fitted; oscillatory differential equations*

I. INTRODUCTION

In this paper, we will investigate the approximate solution of the initial value problem of ordinary differential equations defined on the interval $[x_0, x_{\text{end}}]$ (ODEs):

$$y' = f(x,y), y(x_0) = y_0, \tag{1}$$

where $y \in \mathbb{R}^d$, and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. The ODEs (1) are mathematical models of problems occur in many scientific areas, such as astronomy, quantum mechanics, chemistry and chemical physics etc.

On most occasions, the true solution to (1) arising in applications, is not accessible even though it exists. Therefore it becomes common practice to solve (1) by numerical approaches, among which the classical RK methods are most popular[1] for their desirable characters, such as simple iterative form, high efficiency, and low requirements on the initial values etc. Recently, efforts have been made to suggest adapted-type RK methods in view of the oscillating or periodic solutions to ODEs (1). For example, the exponentially/ trigometrically-fitted RK methods in [2,3], the phase optimized RK methods [4,5] with zero dispersion, or dissipation, or their derivatives.

Taking into consideration high efficiency of RK methods, Chan et al. [6] extended them to incorporate the second derivative, which can achieve high order with few function evaluations. Many variants of the TDRK method have been proposed[7,8].

Here, based on the trigonometrically fitted study on the classical RK method, we propose a new modified TDRK method for solving (1). The rest of this paper is organized as follows. In Sect. 2, we simply recall some relevant basic definitions and properties for TDRK method. In Sect. 3, we construct our modified TDRK step by step, and analyze its phase properties subsequently. In Sect. 4, we provide computational experiments to show its practical performance. Finally, we present remarks and conclusions in Sect. 5.

II. THE MODIFIED TDRK METHOD

Motivated by the TDRK method in [6], we propose the following modified explicit TDRK method:

$$\begin{cases} Y_i = \gamma_i y_n + c_i h f(x_n, y_n) + h^2 \sum_{j=1}^{i-1} a_{ij} g(x_n + c_j h, Y_j), i = 2,\ldots,s, \\ y_{n+1} = y_n + h\beta f(x_n, y_n) + h^2 \sum_{i=1}^{s} b_i g(x_n + c_i h, Y_i) \end{cases} \tag{2}$$

where $a_{ij}, b_i, c_i (1 \le j < i \le s)$, and

$$g(x,y) = y''(x,y) := \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} f(x,y),$$

are some real constants. Obviously, the coefficients of the above iterative scheme can be expressed by the following Butcher tableau:

$$\begin{array}{c|c|cccc} 0 & \gamma_1 & 0 & & & \\ c_2 & \gamma_2 & a_{12} & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ c_s & \gamma_n & a_{s1} & \cdots & a_{s,s-1} & \\ \hline & & b_1 & \cdots & b_{s-1} & b_s \end{array}$$

Compared with the classical TDRK method in [6], the extra frequency-depending parameters $\gamma_i(v), v = h\omega, (i = 1,\ldots,s)$ are incorporated to accommodate the special oscillatory structure of ODEs (1), and $\gamma_i(v), v = h\omega, (i = 1,\ldots,s)$ satisfy the property:

$\lim_{v\to 0}\gamma_i(v)=1$ such that the modified TDRK method (2) can reduce to the classical TDRK method.

Based on [6], the conditions for order up to four are listed as follows:

- Order 2 requires: $\sum_{i=1}^{s} b_i = \frac{1}{2}.$
- Order 3 requires in addition: $\sum_{i=2}^{s} b_i c_i = \frac{1}{6}.$
- Order 4 requires in addition: $\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{12}.$

In addition, the following simplifying assumption is often used:

$$\sum_{j=1}^{s} a_{ij} = \frac{1}{2} c_i^2, \text{ for } i = 2,\cdots,s. \tag{3}$$

For $s=2$, taking $\gamma_1=\gamma_2=1$ in the scheme (2) and solving the equations in orders 2-4 and (3), Chan et al.[6] derived an explicit two stage TDRK method as follows:

0	1	0	0
$\frac{1}{2}$	1	$\frac{1}{8}$	0
		$\frac{1}{6}$	$\frac{1}{3}$

In the next section, we shall derive a new trigonometrically fitted TDRK method with a real frequency-depending parameter $\gamma_2(v)$.

III. The New Trigonometrically Fitted TDRK Method

Now we consider the following modified two-stage TDRK method given by the Butcher tableau:

0	1	0	0
$c_2(v)$	$\gamma_2(v)$	$a_{12}(v)$	0
		$b_1(v)$	$b_2(v)$

In which, $c_2(v),\gamma_2(v),a_{21}(v),b_1(v),b_2(v)$ are some real even functions of $v=h\omega$. Considering that the solution to the problem (1) is often periodic or oscillatory, now we deduce some frequency-depending coefficients by requiring the above modified TDRK to integrating exactly $exp(i\omega x)$, that is

$$\begin{aligned}\exp(\mathrm{i}v) &= \cos(v)+\mathrm{i}\sin(v)\\ &= 1+\mathrm{i}v-(b_1(v)+\gamma_2(v)b_2(v))v^2\\ &\quad -\mathrm{i}c_2(v)b_2(v)v^3+a_{21}(v)b_2(v)v^4,\end{aligned} \tag{4}$$

which gives

$$\begin{aligned}\cos(v) &= 1-(b_1(v)+\gamma_2(v)b_2(v))v^2+a_{21}(v)b_2(v)v^4,\\ \sin(v) &= v-c_2(v)b_2(v)v^3.\end{aligned} \tag{5}$$

Solving Equations in Order 2 and 4, and Equations (3) and (5), we obtain the desired coefficients:

In some cases, the following Taylor series expansions of (7) must be used

$$c_2(v)=\frac{v^3}{12(-\sin(v)+v)},$$

$$\gamma_2(v)=\frac{1}{288}\frac{\varphi(v)}{(\sin(v))^2-2\sin(v)v+v^2},$$

$$a_{21}(v)=\frac{1}{288}\frac{v^6}{(-\sin(v)+v)^2},$$

$$b_1(v)=\frac{-24(\sin(v))^2+48\sin(v)v-24v^2+v^6}{2v^6},$$

$$b_2(v)=12\frac{(\sin(v))^2-2\sin(v)v+v^2}{v^6},$$

where

$$\begin{aligned}\varphi(v) &= v^8-12v^6+24v^4-24\cos(v)v^4\\ &\quad +288v^2-576\sin(v)v+288(\sin(v))^2.\end{aligned}$$

Occasionally, the following Taylor expansions should be used:

$$c_2(v)=\frac{1}{2}+\frac{1}{40}v^2+\frac{11}{16800}v^4+\frac{17}{1512000}v^6+\frac{563}{4656960000}v^8+\cdots;$$

$$\gamma_2(v)=1+\frac{1}{240}v^4+\frac{23}{67200}v^6+\frac{89}{6048000}v^8+\frac{887}{2069760000}v^{10}+\cdots;$$

$$a_{21}(v)=\frac{1}{8}+\frac{1}{80}v^2+\frac{43}{67200}v^4+\frac{19}{864000}v^6+\frac{2071}{3725568000}v^8+\cdots;$$

$$b_1(v)=\frac{1}{6}+\frac{1}{30}v^2-\frac{41}{25200}v^4+\frac{23}{453600}v^6-\frac{157}{139708800}v^8+\cdots;$$

$$b_2(v)=\frac{1}{3}-\frac{1}{30}v^2+\frac{41}{25200}v^4-\frac{23}{453600}v^6+\frac{157}{139708800}v^8+\cdots.$$

The above method is referred as TDRK4NEW. Next we check the order conditions of the new method. In fact, we only need to check order 3 given in Section 2.

Third algebraic order:

$$\begin{aligned}\sum_{i=1}^{2} b_i(v)c_i &= \frac{1}{6}-\frac{1}{120}v^2+\frac{1}{5040}v^4-\frac{1}{362880}v^6+\frac{1}{39916800}v^8+\cdots\\ &= \frac{1}{6}+\mathrm{O}(v^2).\end{aligned}$$

This indicates that our new proposed method is of order four. In addition, after some manipulations, we get the local truncation error of TDRK4NEW:

$$\text{L.T.E.}=\frac{h^5}{17280}(2880y^{(5)}+144y^{(3)}+120g_2y^{(3)})+\mathrm{O}(h^6).$$

Now, we analyze the phase properties of TDRK4NEW.

Definition 1 For the TDRK with stability function $M(\mathrm{i}\theta,v)$, the following two quantities

$$\tilde{P}(\theta,v)=\theta-\arg(M(\mathrm{i}\theta,v)),\tilde{D}(\theta,v)=1-|M(\mathrm{i}\theta,v)|$$

are called the phase-lag (dispersion) and amplification factor error (dissipation), respectively. If

$$\tilde{P}(\theta,v)=c_\varphi\theta^{q+1}+\mathrm{O}(\theta^{q+3}),\tilde{D}(\theta,v)=c_d\theta^{p+1}+\mathrm{O}(\theta^{p+3}),$$

then the corresponding TDRK is said to be of a phase lag order q and dissipation order p, respectively.

Let $r = v/\theta = \omega/\lambda$, and we can deduce the following expressions for the phase-lag and the amplification of TDRK4NEW:

$$\tilde{P}(\theta, r\theta) = \frac{1-r^2}{120}\theta^5 + \frac{1-r^4}{840}\theta^7 + O(\theta^9),$$

$$\tilde{D}(\theta, r\theta) = \frac{(r^2-1)(r^2-5)}{720}\theta^6 - \frac{35+56r^2+20r^4+r^6}{40320}\theta^8 + O(\theta^{10}),$$

Therefore, TDRK4NEW has a phase-lag of order four and a dissipation of order five.

IV. Numerical Results

In this section, we use some available ODEs to illustrate the feasibility and effectiveness of TDRK4NEW. Moreover, we also test against some famous four order RK methods listed as follows:

(i). MTDRKA: The first modified TDRK method derived by Fang et al in [7].

(ii). MTDRKB: The second modified TDRK method derived by Fang et al in [7].

(iii). TDRK4NEW: The new modified TDRK method derived in Section 3.

Problem 1. First we consider the inhomogeneous equation as follows:

$$y'' + 100y = 99\sin(x), y(0) = 1, y'(0) = 11,$$

which has exact solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$. Set $\omega = 10$ and solve the problem numerically on the interval [0,1000]. We list the end-point global error in Table 1, where h denotes the step size.

TABLE I. Comparison of the End-Point Global Errors

h	MTDRKA	MTDRKB	TDRK4NEW
$1/2^7$	5.1005×10^{-5}	1.0174×10^{-4}	6.7096×10^{-10}
$1/2^8$	1.5908×10^{-6}	3.1788×10^{-6}	1.9013×10^{-11}
$1/2^9$	4.9715×10^{-8}	9.9359×10^{-8}	5.5278×10^{-13}
$1/2^{10}$	1.5551×10^{-9}	3.1070×10^{-9}	3.2307×10^{-14}

Problem 2. Now we test the two-dimensional problem:

$$y'' + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} y = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, y'(0) = \begin{pmatrix} -4 \\ 8 \end{pmatrix},$$

where

$$\begin{cases} f_1(x) = 9\cos(2x) - 12\sin(2x), \\ f_2(x) = -12\cos(2x) + 9\sin(2x). \end{cases}$$

The analytical solution is:

$$y(x) = \begin{pmatrix} \sin(x) - \sin(5x) + \cos(2x) \\ \sin(x) + \sin(5x) + \sin(2x) \end{pmatrix}.$$

Here we take $\omega = 5$. The numerical results listed in Table 2 are obtained with the step-size $h = 2^{-2-i}, i = 1,2,3,4$ on the interval [0,100].

TABLE II. Comparison of the End-Point Global Errors

h	MTDRKA	MTDRKB	NETDRK
$1/2^3$	2.2380×10^{-1}	1.5990×10^{-1}	6.0000×10^{-3}
$1/2^4$	1.3300×10^{-2}	1.1800×10^{-2}	4.4470×10^{-4}
$1/2^5$	8.3418×10^{-4}	7.6467×10^{-4}	2.9818×10^{-5}
$1/2^6$	5.2500×10^{-5}	4.8461×10^{-5}	1.9229×10^{-6}

The results in Tables 1-2 indicate that TDRK4NEW is competitive with and even slightly better than the other two methods.

V. Conclusions

In this paper, we proposed, analyzed, and tested a new trigonometrically fitted two-derivative Runge-Kutta method for ODEs with oscillatory solutions. Its local truncation error is also given. Numerical experiments on two typical ODEs illustrated that the proposed method is competitive with some recently designed TDRK methods.

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