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An Algorithm for Dynamic Rigid-Body Model

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Abstract—**In this paper, an algorithm for dynamic rigid-body model is considered. To this end, we first develop an equivalent reformulation of the problem via the Fischer function. Based on this, we propose a new type of method for solving the problem, and the algorithm is shown to be globally convergent and quadratically convergent without nondegenerate assumption.**

Keywords- dynamic rigid-body model; linear complementarity problem ; algorithm; globally convergent ;quadratic convergence

I. INTRODUCTION

In the three-dimensional rigid-body frictional contact model described in [1, 2, 3], there are n_c contact points which are classified as rolling contacts or sliding contacts, a rigid object comes into contact with a number of manipulator links at a finite number of points, and the contact forces obey a Coulomb friction law, and the certain kinematic acceleration constraints, the Signorini nonpenetration condition are also satisfied. Thus, we may write the dynamic rigid-body model as the following linear complementarity problem, abbreviated as LCP ([1]):

$$\begin{bmatrix} a_{Sn} \\ a_{Rn} \\ a_{Rt}^{+} \\ a_{Ro}^{+} \\ \rho_{Rt}^{-} \\ \rho_{Ro}^{-} \end{bmatrix} = M \begin{bmatrix} c_{Sn} \\ c_{Rn} \\ \rho_{Rt}^{+} \\ \rho_{Ro}^{+} \\ a_{Rt}^{-} \\ a_{Ro}^{-} \end{bmatrix} + p,$$

$$\begin{aligned} &\min\{a_{Sn}, c_{Sn}\} = 0, \min\{a_{Rn}, c_{Rn}\} = 0, \\ &\min\{a_{Rt}^{+}, \rho_{Rt}^{+}\} = 0, \min\{a_{Ro}^{+}, \rho_{Ro}^{+}\} = 0, \\ &\min\{a_{Rt}^{-}, \rho_{Rt}^{-}\} = 0, \min\{a_{Ro}^{-}, \rho_{Ro}^{-}\} = 0, \end{aligned} \tag{1}$$

where the subscripts n,t,o denote the three components (normal, tangential, orthogonal) of the accelerations and forces, R and S be two partitioning subsets of $\{1,2,\cdots,n_c\}$ denoting, respectively, the rolling and sliding contacts, U_R being the diagonal matrix with diagonal entries u_i, $i \in R$, the matrix $M \in R^{m\times m}$ and vector $p \in R^m$ contain given data, defined as follows:

$$M = \begin{bmatrix} (M_{nn})_{SS} & (M_{nn})_{SR} & (A_{nt})_{SR} & (A_{no})_{SR} & 0 & 0 \\ (M_{nn})_{RS} & (M_{nn})_{RR} & (A_{nt})_{RR} & (A_{no})_{RR} & 0 & 0 \\ (M_{tn})_{RS} & (M_{tn})_{RR} & (A_{tt})_{RR} & (A_{to})_{RR} & I & 0 \\ (M_{on})_{RS} & (M_{on})_{RR} & (A_{ot})_{RR} & (A_{oo})_{RR} & 0 & I \\ 0 & 2U_R & -I & 0 & 0 & 0 \\ 0 & 2U_R & 0 & -I & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} b_{Sn} \\ b_{Rn} \\ b_{Rt} \\ b_{Ro} \\ 0 \\ 0 \end{bmatrix}.$$

For the ease of description, we denote

$$x = \left(c_{Sn}, c_{Rn}, \rho_{Rt}^{+}, \rho_{Ro}^{+}, a_{Rt}^{-}, a_{Ro}^{-}\right)^T \in R^m .$$

Thus, system (1) can be written as

$$x \ge 0, \quad Mx + p \ge 0, \quad x^{\bullet}\,(Mx + p) = 0. \tag{2}$$

We assume that the solution set of the LCP is nonempty throughout this paper, and denote it by X^*.

In recent years, the LCP has received much attention, and many efficient solution methods have been proposed for solving it ([4]), the basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem. Different from the algorithms listed above. We propose a new type of method for solving the problem in this paper. In detail, we first equivalently reformulate the LCP as a system of nonsmooth equations via the Fischer function in section 2, and the differential property, locally Lipschitzian, and strongly semi-smooth of the merit function are also given. In section 3, based on this reformulation, a method to calculate a generalized Jacobian is given, and the theoretical results that the stationary points of the merit function are the solution of the LCP are also presented. In section 4, we propose a new type algorithm for solving the problem, and show that the algorithm is both globally and quadratically convergent without nondegenerate solution. These results obtained in this paper extend the existing ones for the problem.

We end this section with some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean norm of vector in the space is denoted by $\|\cdot\|$. We use $x \ge 0$ to denote a nonnegative vector $x \in R^m$ if there is no confusion.

II. PRELIMINARY

Now, we formulate the LCP as a system of equations via the Fischer function ([5]) $\varphi : R^2 \to R^1$ defined by

$$\varphi(a,b) = \sqrt{a^2+b^2} - a - b,\ \forall a,b \in R.$$

A basic property of this function is that

$$\varphi(a,b) = 0 \Leftrightarrow a \ge 0, b \ge 0, ab = 0.$$

For arbitrary vectors $a,b \in R^m$, we define a vector-valued function as follows $\Phi(a,b) = (\varphi(a_1,b_1), \varphi(a_2,b_2), \cdots, \varphi(a_n,b_n))^\bullet$. Obviously, $\Phi(a,b) = 0 \Leftrightarrow a \ge 0, b \ge 0, a^\bullet b = 0.$

By (2), we define vector-valued function $\Psi : R^m \to R^m$ and real-valued function $f : R^m \to R$ as follows:

$$\Psi(x) := \Phi(x, Mx+p), \tag{3}$$

$$f(x) := \frac{1}{2}\Psi(x)^\bullet \Psi(x) = \frac{1}{2}\|\Psi(x)\|^2. \tag{4}$$

then the following result is straightforward.

Theorem 1. x^* is a solution of the LCP if and only if $\Psi(x^*) = 0$.

From Theorem 1, we know that a point x^* is a solution of the LCP if and only if x^* is a global minimizer with the objective value zero of the following unconstrained optimization problem

$$\min_{x \in R^m} f(x) \tag{5}$$

A favorable property of the function $f(x)$ is that it is continuously differentiable on the whole space R^n although $\Psi(x)$ is not in general. We summarize the differential properties of Ψ and f defined by (3) and (4). The following properties about the strongly semi-smooth function and locally Lipschitzian are due to [6, 7, 8].

Proposition 1. For the vector-valued function Ψ and real-valued function f defined by (3) and (4). Then, the following statements hold.

(a) Ψ is locally Lipschitzian , i.e.,

$$\|\Psi(x+h) - \Psi(x)\| \le c_1 \|h\|, \tag{6}$$

where $c_1 > 0$ is constant.

(b) Ψ is strongly semi-smooth, i.e., for any $h \to 0$,

$$\|\Psi(x+h) - \Psi(x) - Vh\| \le c_2 \|h\|^2, \tag{7}$$

where $c_2 > 0$ is constant.

(c)□f is continuously differentiable, and its gradient at a point $x \in R^n$ is given by $\nabla f(x) = V^T\Psi(x)$, where V is an arbitrary element belonging to $V \in \partial\Psi(x)$ which denote the Clarke's generalized Jacobian of $\Theta(x)$ at $x \in R^n$ ([9]).

III. STATIONARY POINT AND NONSINGULARITY CONDITIONS

In this section, we present the conditions under which a stationary point of (5) is its a global minimizer with the objective value zero.

First, we present an overestimate of Clarke's generalized Jacobian of $\Psi(x)$. Similar to the discussion of Proposition 3.1 in [10], we have the following result.

Proposition 2. For any $x \in R^m$, we have

$$\partial\Psi(x,y) \subseteq D_a + D_b M,$$

where $D_a = diag(a_1, a_2 \cdots, a_m) \in R^{m\times m}, D_b = diag(b_1, b_2 \cdots, b_m) \in R^{m\times m}$,

$$a_i = \frac{x_i}{\sqrt{x_i^2 + (Mx+p)_i^2}} - 1, b_i = \frac{(Mx+p)_i}{\sqrt{x_i^2 + (Mx+p)_i^2}} - 1, \quad if\ x_i^2 + (Mx+p)_i^2 > 0$$

$$a_i = \xi_i - 1, b_i = \eta_i - 1 \text{ for } (\xi_i, \eta_i) \in R^2 \text{ such that } \|(\xi_i, \eta_i)\| \le 1, \quad if\ x_i^2 + (Mx+p)_i^2 = 0.$$

Combining this with the proof of Theorem 27 in [11], we give the following approach to calculate an element of $\partial\Psi(x)$.

Proposition 3. For any $x \in R^m$, choose $u \in R^m$ such that $u_i \ne 0$ for any index i with $x_i = 0$ and $(Mx+p)_i = 0$, and letting $V = D_a + D_b M$, where

$$a_i = \frac{x_i}{\sqrt{x_i^2 + (Mx+p)_i^2}} - 1, b_i = \frac{(Mx+p)_i}{\sqrt{x_i^2 + (Mx+p)_i^2}} - 1, \quad if\ x_i^2 + (Mx+p)_i^2 > 0,$$

$$a_i = \frac{1}{\sqrt{u_i^2+1}} - 1, \qquad b_i = \frac{u_i}{\sqrt{u_i^2+1}} - 1, \quad if\ x_i^2 + (Mx+p)_i^2 = 0.$$

Then $V \in \partial\Psi(x)$.

For simplicity, based on Proposition 3, we take

$$u_i = \begin{cases} 0 & x_i^2 + (Mx+p)_i^2 > 0, \\ 1 & x_i^2 + (Mx+p)_i^2 = 0. \end{cases}$$

Based on above analysis, the following theorem gives a suitable condition which guarantees that every stationary point is a solution of the LCP. First, we give the needed definition ([12]).

Definition 1. A matrix $M \in R^{m\times m}$ is said to be a P_0-matrix if it satisfies the condition: for each vector $x \ne 0$, there exists an index k such that $x_k \ne 0$ and $x_k(Mx)_k \ge 0$.

Theorem 2. Suppose that matrix $M \in R^{m\times m}$ is P_0-matrix. Then, $V = D_a + D_b M$ is nonsingularity, where V defined in Proposition 3

Proof: Assume that V is not nonsingularity. Then, there exists a nonzero vector $\omega \in R^m$ such that $V\omega = 0$, i.e.,

$$(D_a + D_b M)\omega = 0.$$

By simple algebra yields $a_k\omega_k + b_k(M\omega)_k = 0$. (8)

For any k such that $\omega_k \neq 0$, combining $a_k < 0$ with $b_k < 0$, we have

$$\omega_k(M\omega)_k = -\frac{a_k}{b_k}\omega_k^2 < 0.$$

Combining this with the hypotheses of P_0-matrix, this is contradiction. Thus, the desired result follows.

Based on Theorem 2 and Proposition 1(c), we have the following conclusion which can be easily proved.

Theorem 3. Suppose that matrix $M \in R^{m\times m}$ is P_0-matrix. x^* is a stationary point of (5), then x^* is a solution of the LCP.

IV. Algorithm and Convergence

In this section, a method with Armijo step size rule is presented for solving the LCP, and also discuss its the global convergence and quadratic convergence.

Algorithm 1

Step 1: Choose any point $x_0 \in R^m$, parameters $\sigma, \beta, \gamma \in (0,1)$ and $\varepsilon \geq 0$. Let $k = 0$.

Step 2: If $\|\nabla f(x^k)\| \leq \varepsilon$, stop; Otherwise, go to Step 3.

Step 3: Choose $V^k \in \partial\Psi(x^k)$. Let $d^k \in R^m$ be a solution of the linear system

$$((V^k)^T V^k + \mu^k I)d = -(V^k)^T \Psi(x^k). \quad (9)$$

If d^k satisfies

$$\|\Psi(x^k + d^k)\| \leq \gamma \|\Psi(x^k)\|, \quad (10)$$

then $x^{k+1} = x^k + d^k$, $k := k+1$, go to Step 5. Otherwise, go to Step 4.

Step 4: Let l_k be the smallest non-negative integer l such that

$$f(x^k + \sigma^l d^k) \leq f(x^k) + \beta\sigma^l \nabla f(x^k)^T d^k.$$

Let $x^{k+1} := x^k + \sigma^{l_k} d^k$.

Step 5: Let $\mu^{k+1} = \|\Psi(x^{k+1})\|^2$, $k := k+1$, go to Step 2.

It is easy to verify that d^k is a descent direction of $f(x)$ at x^k and the algorithm is well defined. Obviously, if $\nabla f(x^k) = 0$, then x^k is a stationary point of problem (5). Thus, x^k is a solution of the LCP by Theorem 3.

In the following convergence analysis, we assume that $\varepsilon = 0$ and Algorithm 1 generates an infinite sequence. We can obtain the convergence and quadratic convergence of Algorithm 1.

Theorem 4. Any accumulation point of the sequence $\{x^k\}$ generated by Algorithm 1 is a stationary point of (5).

Proof: Let $x^* \in R^m$ be an accumulation point of the sequence $\{x^k\}$, i.e., there exists an infinite subsequence $K \subseteq \{1,2,\cdots\}$ such that $\{x^k\}_K \to x^*$. Using upper semi-continuity of the subdifferential. Then, the sequence $\{V^k\}_{k\in K}$ is bounded, $\{V^k\}_K \to V^*$, and $\{\mu^k\}_K \to \mu^*$. Thus, we obtain

$$\{(V^k)^T V^k + \mu^k I\}_K \to (V^*)^{\text{T}} V^* + \mu^* I.$$

If $\nabla f(x^*) \neq 0$. Then, $\mu^* = \|\Psi(\omega^*)\| \neq 0$, and $(V^*)^{\text{T}} V^* + \mu^* I$ is positive definite. Let d^* be a solution of the following linear systems

$$((V^*)^{\text{T}} V^* + \mu^* I)d = -(V^*)^T \Psi(x^*) = -\nabla f(x^*).$$

Thus,

$$\nabla f(x^*)^{\text{T}} d^* = -\nabla f(x^*)^{\text{T}} ((V^*)^{\text{T}} V^* + \mu^* I)^{-1} \nabla f(x^*) < 0. \quad (11)$$

Combining (9) with the discussion above, we can obtain $\{d^k\}_K \to d^*$. For infinite subsequence $K \subseteq \{1,2,\cdots\}$, there are the following two case to consider.

Case 1. If there exists an infinite subsequence $K^1 \subseteq K$ such that $\|\Psi(x^{k^1} + d^{k^1})\| \leq \gamma \|\Psi(x^{k^1})\|, \forall k^1 \in K^1$. Then, there exists integer k^0 such that

$$\|\Psi(x^{k^1} + d^{k^1})\| \leq \|\Psi(x^{k^0})\| \gamma^{k^1 - k^0}, \forall k^1 \in K^1.$$

Combining $\gamma \in (0,1)$, and letting $k^1 \to \infty$ yields

$$\|\Psi(x^{k^1})\| \to \|\Psi(x^*)\| = 0,$$

by Theorem 1, we have that x^* is a solution of the LCP. Thus, $\nabla f(x^*) = 0$, this is contradiction.

Case 2. For any infinite subsequence $K^2 \subseteq K$, If (10) does not hold. Let l^* be the smallest nonnegative integer l such that

$$f(x^* + \sigma^l d^*) < f(x^*) + \beta\sigma^l \nabla f(x^*)^{\text{T}} d^*.$$

By the continuity of the function f, for $k \in K$ sufficiently large, we have

$$f(x^k + \sigma^{l^*} d^k) \leq f(x^k) + \beta\sigma^{l^*} \nabla f(x^k)^{\text{T}} d^k.$$

From the stepsize rule of l_k, we know that

$$\begin{aligned} f(x^{k+1}) = f(x^k + \sigma^{l^k} d^k) &\leq f(x^k) + \beta\sigma^{l^k} \nabla f(x^k)^{\text{T}} d^k \\ &\leq f(x^k) + \beta\sigma^{l^*} \nabla f(x^k)^{\text{T}} d^k. \end{aligned} \quad (12)$$

Since the sequence $\{f(x^k)\}$ is decreasing and bounded from $\{x^k\}_K \to x^*$, so $\lim_{k\to\infty} f(x^k) = f(x^*)$. Taking the limit on both side of (12), we get $f(x^*) \leq f(x^*) + \beta\sigma^{l^*} \nabla f(x^*)^{\text{T}} d^*$, i.e., $\nabla f(x^*)^{\text{T}} d^* > 0$, this contradicts (11).

Combining Case 1 with Case 2, then the desired result follows.

By Theorem 3 and 4, we immediately obtain the following conclusion.

Theorem 5. Suppose that matrix $M \in R^{m\times m}$ is P_0-matrix.

Then, any accumulation point of the sequence $\{x^k\}$ generated by Algorithm 1 is a solution of the LCP.

To obtain the quadratical convergence of Algorithm 1, we first present the following lemma.

Lemma 1. Given positive constant ρ, then there exists constant $c_3 > 0$ such that

$$dist(x, X^*) \le c_3 r(x), \quad \forall \|x\| \le \rho. \tag{13}$$

where $dist(x, X^*)$ denotes the distance between the point $x \in R^m$ and the solution set X^*, $r(x) = \| \min\{x, Mx + p\} \|$.

Proof Assume that the theorem is false. Then, there exist positive sequence $\{\tau_k\}$ and sequence $\{x^k\}$ such that $\| x^k \| \le \rho$, $\tau_k \to \infty$ as $k \to \infty$ and $dist(x^k, X^*) > \tau_k r(x^k)$. Thus,

$$\frac{r(x^k)}{dist(x^k, X^*)} > \tau_k \to 0 \text{ as } k \to \infty. \tag{14}$$

Since $\{x^k\}$ is bounded and $r(x)$ is continuous, by (14), we have $r(x^k) \to 0$ as $k \to \infty$. Since $\{x^k\}$ is bounded again, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{k_i \to \infty} x^{k_i} = \bar{x}$ with $r(\bar{x}) = 0$. Hence, $\bar{x} \in X^*$. By (14), we can also obtain

$$\lim_{k_i \to \infty} \frac{r(x^{k_i})}{\| x^{k_i} - \bar{x} \|} \le \lim_{k_i \to \infty} \frac{r(x^{k_i})}{dist(x^{k_i}, X^*)} = 0. \tag{15}$$

On the other hand, since x and $Mx + p$ are both polynomial functions with powers 1, respectively, from $x^{k_i} \to \bar{x}$, and $r(x^{k_i}) \to 0$, we know that, for all sufficiently large k_i,

$$\frac{r(x^{k_i})}{\| x^{k_i} - \bar{x} \|} \ge \theta,$$

for some positive number θ, this contradicts (15). Thus, (13) holds.

In this following, allows us to extend above this error bound in Lemma 1 to another residual function $\Psi(x)$. First, we give result in which Tseng [13] showed.

Lemma 2. For any $(a, b) \in R^2$, we have
$(2 - \sqrt{2}) | \min\{a, b\} | \le | \phi(a, b) | \le (\sqrt{2} + 2) | \min\{a, b\} |$.

By Lemma 1 and 2, it is easy to deduce the following conclusion.

Proposition 4 Given positive constant ρ, there exists a constant $\eta > 0$ such that

$$dist(x, X^*) \le \eta \| \Psi(x) \|, \forall \| x \| \le \rho. \tag{16}$$

Based on above analysis, by (6), (7) and (16), using the similar technique to that of the proof of Theorem 3.1 in [14], combining Theorem 5, we can obtain the quadratical convergence of Algorithm 1.

Theorem 6. Suppose that matrix $M \in R^{m \times m}$ is P_0-matrix, letting $\{x^k\}$ be generated by Algorithm 1. Then, $dist(x^k, X^*)$ converges to 0 quadratically.

In Theorem 6, we have showed that Algorithm 1 has a quadratic rate of convergence under P_0-matrix, it is an extensions of the algorithm converges conclusion in [15], which is a new result for this problem.

V. CONCLUSIONS

In this paper, we consider an algorithm for solving dynamic rigid-body linear complementarity problem (LCP) model. First, the LCP is reformulated as a system of nonsmooth equations via the Fischer function, and then we propose a new type of method, and the algorithm is shown to be globally convergent and quadratically convergent without nondegenerate assumption. Moreover, Theoretical results that relate the stationary points of the merit function to the solution of the LCP are also presented. However, it would be interesting to investigate whether the algorithm convergent results for LCP hold is strictly weaker than those existing ones. These will be our further research directions.

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