Adaptive Phase Matching in Grover's algorithm with Weighted Targets

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Abstract—**When the Grover's algorithm is applied to search an unordered database, the difference in marked items is not taken into consideration. When the fraction of marked items is greater than 1/4, the successful probability rapidly decreases with the increase of marked items, and when the fraction of marked items is greater than 1/2, the algorithm is disabled. Aiming at these two problems, first, an improved Grover's algorithm with the weighted targets is proposed in which each target is endowed a weight coefficient according to its significance. Using these weight coefficients, we rewrite the targets as a quantum superposition, which can make the probability for getting each target is approximately equal to its weight coefficient. Secondly, the adaptive phase matching is proposed based on the weighted targets, in which the directions of two phase rotations are contrary, and the amplitudes of two phase rotations are determined by the inner-product of the target quantum superposition and the initial state of system. When this inner-product is greater than 0.3090, the successful probability is equal to 1 with at most two Grover iterations. The validity of the improved quantum searching algorithm and the new phase matching are verified by two search examples.**

Keywords- quantum computing; quantum searching; Grover's algorithm; weighted target; phase matching

I. INTRODUCTION

The quantum computing is a rapidly rising frontier that performs information processing by the quantum mechanics. The research of quantum computing in recent 20 years shows that the quantum computing is much better than the classical counterpoints. In the aspect of solving actual problem by using the quantum parallelism, many quantum algorithms have been proposed in which Grover's quantum searching algorithm [1] and Shor's quantum algorithm of very large integer factorization [2] are the most famous. The Grover's algorithm has such two features as follows: firstly, for searching a marked state in an unordered database, it achieves quadratic speed up substantially over many (though not all) classical search algorithms that use search heuristics. Secondly, rather than search the elements directly, Grover's algorithm concentrates on index to those elements, which is just a number in the range 0 to N-1. Hence, the Grover's algorithm is general. As the two features mentioned above, the Grover's algorithm has been widely paid attention. The research of Grover's algorithm has been concentrated on improving, generalizing, and applying since it was proposed in 1996. In the aspect of improving, Grover argued that, in [3], the Hadamard transform, used in the original setting, might be replaced by an arbitrary unitary transformation. Long presented the new phase matching [4], in which two inversions (namely, rotations with π) are replaced by two arbitrary phase rotations that is equal to each other. In the aspect of generalizing, a generalization was obtained by allowing the replacement of the uniform superposition of all basis states, used as the initial state of the algorithm in the original setting, by an arbitrary pure [5] or mixed [6] quantum state. In the aspect of applying, a quantum associative memory model [7] was presented by combining the Grover's algorithm with neural network, which takes on an exponential increase in the capacity of the memory when compared to traditional associative memories such as the Hopfield network. In [8], a quantum associative memory with distributed queries was proposed by generalizing the model mentioned above.

Grover's algorithm provides a quantum method for solving unstructured search problems in roughly the square root of the number of steps required using a classical computer. This amounts to a polynomial speed up over what is possible classically. Although this is not as impressive a speedup as that seen in other quantum algorithms, such as the Deutsch-Jozsa algorithm, for which an exponential speedup is obtained, Grover's algorithm is applicable to a much wider range of computational problems. Moreover, a quadratic speedup is not bad either. While it won't tame problems having an exponential complexity scaling it could, nevertheless, allow significantly larger problem instances to be solved than might otherwise be possible. For example, in an airline scheduling problem any given airline only has finitely many aircraft, and finitely many routes. It is quite possible that a quadratic speedup in solving a scheduling problem is sufficient to confer a practical advantage.

The problem of the current Grover's quantum searching algorithm is below. First, the search targets are treated equally without discrimination, the difference in importance of the targets is not considered, and the probability of getting each target is equal, which is out of place in some cases. Secondly, when the fraction of marked items is greater than 1/4, the successful probability rapidly decreases with the increase of marked items, and when the fraction of marked items is greater than 1/2, the algorithm is disabled. To solve these problems, this paper proposes an improved Grover's searching algorithm that based on the weighted targets. Firstly, each search target is endowed a weight coefficient. Secondly, all search targets are represented as a quantum superposition state using weight coefficient endowed, which shows the difference in importance of each search target. Based on the quantum superposition of the search targets, an improved Grover's algorithm is presented in which the probability of finding each target is approximately equal to its weight coefficient. At the same time, we propose a new phase matching based on the weighted targets. When the inner-product of the target superposition and the initial state of system is greater than 0.3090, the marked state is obtained with certainty with at most two Grover iterations.

II. The Grover's Algorithm Based on the Weighted Targets

Although the original Grover's algorithm achieves quadratic speed up substantially over many classical search algorithms, a certain queries can not be exactly described because all search targets are treated equally without discrimination. Hence, it is necessary to generalize the original Grover's algorithm to the one based on the weighted targets.

A. Construction of the Quantum Superposition State of the Weighted Targets

Suppose the $|q_1\rangle, |q_2\rangle, \cdots, |q_M\rangle$ denote the marked states. The weight coefficients $w_{q_1}, w_{q_2}, \cdots, w_{q_M}$ denote the important degree of each marked state and satisfy $w_{q_1}+w_{q_2}+\cdots+w_{q_M}=1$ and $w_{q_i}>0$. Let $\Omega=\{q_1,q_2,\cdots,q_M\}$, by regarding the weight coefficient as its probability amplitude, the quantum superposition state of all marked states can be written as

$$|q\rangle=\sum_{i=0}^{N-1} b_i\,|i\rangle=\begin{cases}\sum\sqrt{w_i}\,|i\rangle & i\in\Omega\\ 0\,|i\rangle & i\notin\Omega\end{cases}. \tag{1}$$

Based on Eq.(1), the Oracle operator can be written as

$$O=I-2|q\rangle\langle q|. \tag{2}$$

The Hadamard transformation operator is same as the operator defined in common Grover's algorithm, as follows

$$U=H^{\otimes n}(2|0\rangle\langle 0|-I)H^{\otimes n}, \tag{3}$$

where $H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, amd I is an identity matrix.

B. The Iterative Equation of Algorithm

The iterative equation of algorithm can be obtained using method in [8]. Suppose the algorithm begins with an uniform superposition state $|\phi\rangle=|0\rangle^{\otimes n}=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$, and after t iterations the state of the system is described as $|\phi\rangle^{(t)}=\sum_{i=0}^{N-1}a_i^{(t)}|i\rangle$. Then after the Oracle operator O (first sub-step of an iteration), the superposition becomes

$$|\phi\rangle^{(t+1/2)}=\sum_{i=0}^{N-1}(a_i^{(t)}-2\langle q|\phi\rangle^{(t)}b_i)\,|i\rangle. \tag{4}$$

After the unitary operator U (second sub-step of an iterative), the superposition becomes

$$|\phi\rangle^{(t+1)}=\sum_{i=0}^{N-1}[2\langle a_i^{(t)}-2\langle q|\phi\rangle^{(t)}b_i\rangle-a_i^{(t)}+2\langle q|\phi\rangle^{(t)}b_i]\,|i\rangle, \tag{5}$$

where the $\langle\phi\rangle$ denotes the average of the probability amplitude of all basic states in the $|\phi\rangle$.

Hence, after one Grover iteration, the iterative equation of the probability amplitude of each basic state is taken on

$$a_i^{(t+1)}=2\langle\phi\rangle^{(t)}-4\langle q|\phi\rangle^{(t)}\langle q\rangle-a_i^{(t)}+2\langle q|\phi\rangle^{(t)}b_i. \tag{6}$$

From Eqs.(1, 5), using some algebra gives

$$\langle q|\phi\rangle^{(t+1)}+\langle q|\phi\rangle^{(t-1)}=2\langle q|\phi\rangle^{(t)}(1-2N/\langle q\rangle^2). \tag{7}$$

To obtain the expression of $\langle q|\phi\rangle^{(t)}$, suppose

$$\langle q|\phi\rangle^{(t)}=A\sin(\omega t+\varphi). \tag{8}$$

By inserting this expression into Eq.(7) and using some trigonometry, three parameters in Eq.(8) may be written as

$$\begin{cases}A=1\\ \omega=2\arcsin(\langle q|\phi\rangle)\\ \varphi=\arcsin(\langle q|\phi\rangle)\end{cases}. \tag{9}$$

From Eq.(6), using some trigonometry it is possible to derive

$$a_i^{(t)}=A_i\sin(\omega t+\delta_i), \tag{10}$$

where $A_i=\sqrt{\frac{b_i^2-2b_i\langle q\rangle+1/N}{1-N\langle q\rangle^2}}$, $\delta_i=\arccos\left(\frac{b_i-\langle q\rangle}{\sqrt{b_i^2-2b_i\langle q\rangle+1/N}}\right)$.

C. The Successful Probability of Algorithm

By Eq.(8), Eq.(7) may be represented as follows

$$\langle q|\phi\rangle^{(t)} = \sin(\omega t + \varphi) \ . \qquad (11)$$

If the initial state $|\phi\rangle$ of the system is equal to the superposition state $|q\rangle$ after some Grover iterations, the successful probability should be equal to 1. Considering $\langle q|\phi\rangle = 1$, hence, it is necessary to study the relation between the successful probability P and the inner product $\langle q|\phi\rangle$. For this relation, we propose such theorem as follows.

Theorem 1: After t Grover iterations, the successful probability satisfy such relation as follows

$$P^{(t)} \geq (\langle q|\phi\rangle^{(t)})^2 \ . \qquad (12)$$

Proof

$$P^{(t)} = \sum_{i\in\Omega}(a_i^{(t)})^2 = \left(\sum_{i\in\Omega}(a_i^{(t)})^2\right)\left(\sum_{i\in\Omega}b_i^2\right)$$

$$\geq \left(\sum_{i\in\Omega}a_i^{(t)}b_i\right)^2 = \left(\sum_{i=0}^{N-1}a_i^{(t)}b_i\right)^2 = \left(\langle q|\phi\rangle^{(t)}\right)^2 . \qquad \square$$

Hence, it is sufficient that the successful probability is expressed by the square of the inner product of the $|\phi\rangle$ and the $|q\rangle$ after some Grover iterations. Noting that, the parameter t denotes the iteration steps, hence, after t Grover iterations, the successful probability can be obtained by Eq.(11).

Noting that, when $t = \frac{\pi/2 - \varphi}{\omega}$, $\left(\langle q|\phi\rangle^{(t)}\right)^2 = 1$. However, at this time, t usually is not an integer, the iteration steps may be obtained by

$$t_0 = \left[\frac{\pi/2 - \varphi}{\omega}\right] = \left[\frac{\arccos(\langle q|\phi\rangle)}{2\arcsin(\langle q|\phi\rangle)}\right], \qquad (13)$$

wher $[\bullet]$ denotes the rounding operator.

Let $\lambda = (\langle q|\phi\rangle)^2$, $(0 \leq \lambda \leq 1)$, then,

$$t_0 = \left[\frac{\arccos(\sqrt{\lambda})}{2\arcsin(\sqrt{\lambda})}\right]. \qquad (14)$$

The curve of $(\langle q|\phi\rangle^{(t_0)})^2$ is shown in Fig.1.

It can be seen from Fig.1 that, after at most t_0 iterations, observation of the state in the computational basis then yields a solution to the search problem with probability at least one-half. In fact, the state $|\phi\rangle$ of the system is evolved toward the target superposition state $|q\rangle$ in this algorithm. Therefore, after t_0 iterations, the probability of each target is approximately equal to its weight coefficient, which the importance of each target is shown by its probability. However, the following problems remain, namely, the successful probability decreased rapidly as the λ increases.

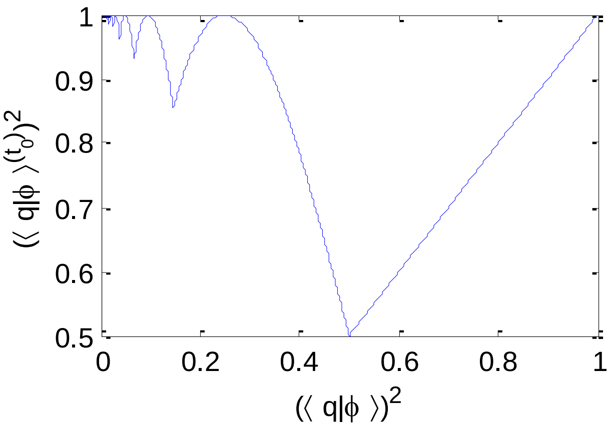

Figure 1. The probability curve of the Grover's algorithm based on the weighted targets.

D. The Relation Between the Weighted Grove's Algorithm and the Original one

For the relation between the improved Grover's algorithm and the original one, we have such conclusion as follows.

Theorem 2: The original Grover's algorithm is a particular case of the improved one, and when all weight coefficients are equal in the improved Grover's algorithm, these two algorithms is equivalent.

Proof Let the target superposition state in the improved algorithm is taken on

$$|q\rangle = \sum_{i=1}^{M} b_i |q_i\rangle = \sum_{i=1}^{M}\sqrt{w_i}\,|q_i\rangle \ .$$

When $w_1 = w_2 = \cdots = w_M$, it can be derived from Eq.(1) that $b_1 = b_2 = \cdots = b_M = 1/\sqrt{M}$. Hence, $\langle q|\phi\rangle = M\frac{1}{\sqrt{M}}\frac{1}{\sqrt{N}} = \sqrt{M/N}$,

$$t_0 = \left[\frac{\arccos(\langle q|\phi\rangle)}{2\arcsin(\langle q|\phi\rangle)}\right] = \left[\frac{\arccos(\sqrt{M/N})}{2\arcsin(\sqrt{M/N})}\right] = R ,$$

where R is the iteration steps in the original Grover's algorithm. Thus, the iteration steps of these two algorithms are equal. Noting

$$\begin{aligned}(\langle q|\phi\rangle^{t_0})^2 &= \sin^2(2\arcsin(\langle q|\phi\rangle)t_0 + \arcsin(\langle q|\phi\rangle)) \\ &= \sin^2((2t_0+1)\arcsin(\langle q|\phi\rangle)) \\ &= \sin^2((2t_0+1)\arcsin(\sqrt{M/N})) = P\end{aligned} ,$$

where P is the probability of finding the targets in the original Grover's algorithm. Therefore, after the same iteration steps, the probabilities of two algorithms are equal, namely, these two algorithms are equivalent. □

III. The Adaptive Phase Matching for Weighted Grover's Algorithm

A. The Adaptive Phase Matching

The two phase shift operators in the weighted Grover's algorithm may be generally expressed as follows

$$O = I - (1-e^{i\alpha})|q\rangle\langle q|, \tag{15}$$

$$U = (1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I\,. \tag{16}$$

The adaptive phase matching proposed in this paper can be expressed as the following theorem.

Theorem 3: Let $\lambda = \langle q|\phi\rangle^2$.

(1) When $\begin{cases} 1/4 \le \lambda < 1 \\ \alpha = -\beta = \arccos(\frac{2\lambda-1}{2\lambda}) \end{cases}$, the successful probability $P=1$ can be obtained after only one iteration.

(2) When $\begin{cases} \frac{3-\sqrt{5}}{8} < \lambda < \frac{1}{4} \\ \alpha = -\beta = \arccos(1-\frac{3-\sqrt{5}}{4\lambda}) \end{cases}$, the successful probability $P=1$ can be obtained after only two iterations.

Proof

(1) $|\phi\rangle^{(1/2)} = O|\phi\rangle = [I-(1-e^{i\alpha})|q\rangle\langle q|]|\phi\rangle$

$$= |\phi\rangle - (1-e^{i\alpha})\langle q|\phi\rangle|q\rangle\,,$$

$$|\phi\rangle^{(1)} = U|\phi\rangle^{(1/2)}$$

$$= [(1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I]|\phi\rangle^{(1/2)}$$

$$= |\phi\rangle - (1-e^{i\alpha})(1-e^{i\beta})\langle q|\phi\rangle^2 - e^{i\beta}(1-e^{i\alpha})\langle q|\phi\rangle|q\rangle\,,$$

$$\langle q|\phi\rangle^{(1)} = -(1-e^{i\alpha})(1-e^{i\beta})\langle q|\phi\rangle^3 + (1-e^{i\beta}(1-e^{i\alpha}))\langle q|\phi\rangle\,.$$

Let $\alpha = -\beta$, inserting $\lambda = \langle q|\phi\rangle^2$ in $\langle q|\phi\rangle^{(1)}$ and using some algebra it is possible to derive

$$|\langle q|\phi\rangle^{(1)}|^2 = (4\lambda^3-4\lambda^2)\cos^2\alpha - (8\lambda^3-12\lambda^2+4\lambda)\cos\alpha + (4\lambda^3-8\lambda^2+5\lambda)\,.$$

When $\cos\alpha = (2\lambda-1)/(2\lambda)$, namely, $\alpha = -\beta = \arccos((2\lambda-1)/(2\lambda))$, $P = |\langle q|\phi\rangle^{(1)}|^2_{\max} = 4\lambda^3 - 8\lambda^2 + 5\lambda - \frac{(8\lambda^3-12\lambda^2+4\lambda)^2}{4(4\lambda^3-4\lambda^2)} = 1$. From $|\cos\alpha| \le 1$, the range $1/4 \le \lambda < 1$ may be obtained.

(2) $|\phi\rangle^{(3/2)} = [I-(1-e^{i\alpha})|q\rangle\langle q|]|\phi\rangle^{(1)}$

$$= |\phi\rangle - (1-e^{i\alpha})(1-e^{i\beta})\langle q|\phi\rangle^2|\phi\rangle - (1+e^{i\beta})(1-e^{i\alpha})\langle q|\phi\rangle|q\rangle + (1-e^{i\alpha})^2(1-e^{i\beta})\langle q|\phi\rangle^3|q\rangle + e^{i\beta}(1-e^{i\alpha})^2\langle q|\phi\rangle|q\rangle\,,$$

$$\begin{aligned} |\phi\rangle^{(2)} &= [(1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I]|\phi\rangle^{(3/2)} \\ &= |\phi\rangle - (1-e^{i\alpha})(1-e^{i\beta})^2\langle q|\phi\rangle^2|\phi\rangle \\ &\quad - (1+2e^{i\beta})(1-e^{i\alpha})(1-e^{i\beta})\langle q|\phi\rangle^2|\phi\rangle \\ &\quad + (1-e^{i\alpha})^2(1-e^{i\beta})^2\langle q|\phi\rangle^4|\phi\rangle \\ &\quad + e^{i\beta}(1-e^{i\alpha})^2(1-e^{i\beta})\langle q|\phi\rangle^2|\phi\rangle \\ &\quad - (e^{i2\beta}+e^{i\beta})(1-e^{i\alpha})\langle q|\phi\rangle|q\rangle \\ &\quad + e^{i\beta}(1-e^{i\alpha})^2(1-e^{i\beta})\langle q|\phi\rangle^3|q\rangle \\ &\quad + e^{i2\beta}(1-e^{i\alpha})^2\langle q|\phi\rangle|q\rangle \end{aligned},$$

$$\begin{aligned} \langle q|\phi\rangle^{(2)} &= \langle q|\phi\rangle - (1-e^{i\alpha})(1-e^{i\beta})^2\langle q|\phi\rangle^3 \\ &\quad - (1+2e^{i\beta})(1-e^{i\alpha})(1-e^{i\beta})\langle q|\phi\rangle^3 \\ &\quad + (1-e^{i\alpha})^2(1-e^{i\beta})^2\langle q|\phi\rangle^5 \\ &\quad + 2e^{i\beta}(1-e^{i\alpha})^2(1-e^{i\beta})\langle q|\phi\rangle^3 \\ &\quad - (e^{i2\beta}+e^{i\beta})(1-e^{i\alpha})\langle q|\phi\rangle + e^{i2\beta}(1-e^{i\alpha})^2\langle q|\phi\rangle \end{aligned}.$$

Let $\alpha = -\beta$, inserting $\lambda = \langle q|\phi\rangle^2$ in $\langle q|\phi\rangle^{(2)}$ and using some algebra it is possible to derive

$$\begin{aligned} |\langle q|\phi\rangle^{(2)}|^2 &= (16\lambda^5-16\lambda^4)\cos^4\alpha - (65\lambda^5-112\lambda^4+48\lambda^3)\cos^3\alpha \\ &\quad + (96\lambda^5-240\lambda^4+188\lambda^3-44\lambda^2)\cos^2\alpha \\ &\quad - (64\lambda^5-208\lambda^4+232\lambda^3-100\lambda^2+12\lambda)\cos\alpha \\ &\quad + (16\lambda^5-64\lambda^4+92\lambda^3-56\lambda^2+13\lambda) \end{aligned}.$$

When $\alpha = -\beta = \arccos(1-(3-\sqrt{5})/(4\lambda))$, $P = |\langle q|\phi\rangle^{(2)}|^2_{\max} = 1$. From $|\cos\alpha| \le 1$, the range $(3-\sqrt{5})/8 < \lambda < 1/4$ may be obtained. □

The successful probability curve is shown in Fig.2, where $\theta_1 = \arccos(1-1/(2\lambda))$, $\theta_2 = \arccos(1-(3-\sqrt{5})/(4\lambda))$.

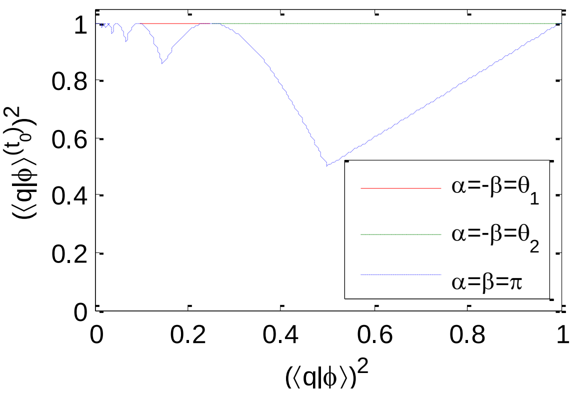

Figure 2. Comparison of successful probability curves between the adaptive phase matching and the original one.

B. Algorithm Description

According to the value of λ, the algorithm implementation steps can be divided into the following three cases.

(I) When $0<\lambda<\frac{3-\sqrt{5}}{8}$, the original phase matching is applied.

(II) When $\frac{3-\sqrt{5}}{8}<\lambda<\frac{1}{4}$, the adaptive phase matching is applied, and $\alpha=-\beta=\arccos(1-\frac{3-\sqrt{5}}{4\lambda})$. The search process can be described as follows:

Step1 Applying Eq.(15) to the initial state of the system, $|\hat{\phi}_1\rangle=O|\phi\rangle$.

Step2 Applying Eq.(16) to the $|\hat{\phi}_1\rangle$, namely, $|\bar{\phi}_1\rangle=U|\hat{\phi}_1\rangle$.

Step3 Reapplying Eq.(15) to the $|\bar{\phi}_1\rangle$, namely, $|\hat{\phi}_2\rangle=O|\bar{\phi}_1\rangle$.

Step4 Reapplying Eq.(16) to the $|\hat{\phi}_2\rangle$, namely, $|\bar{\phi}_2\rangle=U|\hat{\phi}_2\rangle$.

Step5 Measuring $|\bar{\phi}_2\rangle$.

(III) When $1/4<\lambda<1$, the adaptive phase matching is applied, and $\alpha=-\beta=\arccos(\frac{2\lambda-1}{2\lambda})$. The search process can be described as follows:

Step1 Applying Eq.(15) to the initial state of the system, $|\hat{\phi}_1\rangle=O|\phi\rangle$.

Step2 Applying Eq.(16) to the $|\hat{\phi}_1\rangle$, namely, $|\bar{\phi}_1\rangle=U|\hat{\phi}_1\rangle$.

Step3 Measuring $|\bar{\phi}_1\rangle$.

IV. SEARCHING EXAMPLE

A. Weighted Targets Searching

Suppose the system has three qubits and the initial state be a uniform superposition state. The marked states are $|010\rangle$, $|100\rangle$, and $|110\rangle$. The weight coefficients are 0.005, 0.045, and 0.95, respectively. The initial state of the system is expressed as follows

$$|\phi\rangle=\frac{1}{\sqrt{8}}\begin{pmatrix}|000\rangle+|001\rangle+|010\rangle+|011\rangle+\\|100\rangle+|101\rangle+|110\rangle+|111\rangle\end{pmatrix}.$$

By Eq.(1), the superposition of three marked states is

$$|q\rangle=\sqrt{0.005}|010\rangle+\sqrt{0.045}|100\rangle+\sqrt{0.95}|110\rangle.$$

By Eqs.(9, 13), the number of iteration steps is

$$t_0=\left[\frac{\pi/2-\varphi}{\omega}\right]=\left[\frac{\pi/2-0.460729}{0.921458}\right]=1.$$

By Eq.(10), the iterative equation of probability amplitude of each marked state is represented as follows

$$\begin{cases}a_0^t=A_0^t\sin(\omega t+\delta_0)=0.366499\sin(0.921458t+1.837372)\\a_1^t=A_1^t\sin(\omega t+\delta_1)=0.358835\sin(0.921458t+1.399017)\\a_2^t=A_2^t\sin(\omega t+\delta_2)=0.978742\sin(0.921458t+1.369589)\end{cases}.$$

Hence, the probability of getting each search target is

$$\begin{cases}P_{|101\rangle}=a_0^2=(A_0^{t_0}\sin(\omega t_0+\delta_0))^2=0.019737\\P_{|100\rangle}=a_1^2=(A_1^{t_0}\sin(\omega t_0+\delta_1))^2=0.068977\\P_{|100\rangle}=a_2^2=(A_2^{t_0}\sin(\omega t_0+\delta_2))^2=0.884903\end{cases}.$$

The relation between these three successful probabilities and the corresponding weight coefficients is shown in Table1.

TABLE I. THE RELATION BETWEEN THE SUCCESSFUL PROBABILITY AND THE WEIGHT COEFFICIENT

Marked State	$\vert 010\rangle$	$\vert 100\rangle$	$\vert 110\rangle$
Probability	0.018737	0.068977	0.884903
Weight coefficient	0.005000	0.045000	0.950000

It can be seen from Table1 that the probabilities of three marked states fully represent their respective importance, and the more important the target is, the greater probability is obtained, which the flexibility of this algorithm is shown.

According to the above results, when $t_0=1$, the probability of finding all marked states is $P=P_{|010\rangle}+P_{|100\rangle}+P_{|110\rangle}=0.972617$. By Eqs.(8, 9), $\langle q|\phi\rangle^{(t)}=\sin(\omega t+\varphi)=\sin(0.921458t+0.460729)$. We can get $(\langle q|\phi\rangle^{(t_0)})^2=0.964846$ from the above equation. Hence, $P\geq(\langle q|\phi\rangle^{(t_0)})^2$, which is coincident with the conclusion of the **theorem 1**.

For ease of comparison, the searching process by the original Grover's algorithm is presented as follows. In this example, $N=8$, $M=3$, the iteration steps is $R=\left[\frac{\arccos(\sqrt{M/N})}{2\arcsin(\sqrt{M/N})}\right]=1$. The searching process is below

$$|\hat{\phi}\rangle=(I-2\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|)|\phi\rangle$$
$$=\frac{1}{2\sqrt{2}}(1\quad 1\quad -1\quad 1\quad -1\quad 1\quad -1\quad 1)',$$

$$|\bar{\phi}\rangle=(2|\phi\rangle\langle\phi|-I)|\hat{\phi}\rangle=\frac{1}{4\sqrt{2}}(-1\quad -1\quad 3\quad -1\quad 3\quad -1\quad 3\quad -1).$$

The successful probability of each marked state is equal to $P_{|010\rangle}=P_{|100\rangle}=P_{|110\rangle}=\left(\frac{3}{4\sqrt{2}}\right)^2=0.28125$. The total probability is equal to $P=P_{|010\rangle}+P_{|100\rangle}+P_{|110\rangle}=0.84375$, which shows the original Grover's algorithm can not distinguish the importance of different targets.

B. Weighted Targets Searching With Adaptive Phase Matching

There are 32 students in a class whose serial numbers are in range 0 to 31. (I) The search targets are the students whose serial number satisfies $n = 2k+4$, where $k = 0,1,2,\cdots,13$. The target serial numbers and the marked states are shown in Table 2. (II) The search targets are the students whose serial number satisfies $n = 6k+3$, where $k = 0,1,2,\cdots,5$. The target serial numbers and the marked states are shown in Table 3.

TABLE II. THE TARGET SERIAL NUMBERS AND MARKED STATES

k	Serial Number	Marked States	k	Serial Number	Marked States
0	4	$\lvert 00100\rangle$	7	18	$\lvert 10010\rangle$
1	6	$\lvert 00110\rangle$	8	20	$\lvert 10100\rangle$
2	8	$\lvert 01000\rangle$	9	22	$\lvert 10110\rangle$
3	10	$\lvert 01010\rangle$	10	24	$\lvert 11000\rangle$
4	12	$\lvert 01100\rangle$	11	26	$\lvert 11010\rangle$
5	14	$\lvert 01110\rangle$	12	28	$\lvert 11100\rangle$
6	16	$\lvert 10000\rangle$	13	30	$\lvert 11110\rangle$

TABLE III. THE TARGET SERIAL NUMBERS AND MARKED STATES

k	Serial Number	Marked States	k	Serial Number	Marked States
0	3	$\lvert 00011\rangle$	3	21	$\lvert 10101\rangle$
1	9	$\lvert 01001\rangle$	4	27	$\lvert 11011\rangle$
2	15	$\lvert 01111\rangle$			

In these two searching, $N = 32$, using 5 qubits can store all serial numbers. The initial state of the $|\phi\rangle$ is expressed as $|\phi\rangle = \frac{1}{4\sqrt{2}}(|0\rangle + |1\rangle + |2\rangle + \cdots + |31\rangle)$.

(I) Suppose the quantum superposition state of all marked states is constructed as follows

$$|q\rangle = \sqrt{\frac{4}{32}}(|4\rangle + |6\rangle + |8\rangle + |10\rangle + |12\rangle + |14\rangle) + \sqrt{\frac{1}{32}}\begin{pmatrix}|16\rangle + |18\rangle + |20\rangle + |22\rangle + \\ |24\rangle + |26\rangle + |28\rangle + |30\rangle\end{pmatrix},$$

$$\lambda = \langle q|\phi\rangle^2 = \left(\frac{6}{\sqrt{32}}\sqrt{\frac{4}{32}} + \frac{8}{\sqrt{32}}\sqrt{\frac{1}{32}}\right)^2 = \frac{25}{64} > \frac{1}{4},$$

$$\alpha = -\beta = \arccos(1 - 1/(2\lambda)) = \arccos(-7/25).$$

According to **theorem 3**, the probability of getting correct results is equal to 1 only one Grover iteration. The search process can be described as follows.

(1) $|\hat{\phi}\rangle = (I - (1-e^{i\alpha})|q\rangle\langle q|)|\phi\rangle$

$$= \frac{1}{32}\sum_{j0=1}^{17}|q_{j0}\rangle + \left(\frac{1}{\sqrt{32}} - \frac{5}{8}\sqrt{\frac{4}{32}}(1-e^{i\alpha})\right)\sum_{j1=0}^{5}|q_{j1}\rangle + \left(\frac{1}{\sqrt{32}} - \frac{5}{8}\sqrt{\frac{1}{32}}(1-e^{i\alpha})\right)\sum_{j2=0}^{7}|q_{j2}\rangle.$$

(2) $|\bar{\phi}\rangle = ((1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I)|\hat{\phi}\rangle$

$$= \begin{pmatrix} aaaa & baba & baba & baba \\ caca & caca & caca & caca \end{pmatrix},$$

where
$$\begin{cases} a = \frac{1}{\sqrt{32}} - \frac{25(1-\cos\alpha)}{32\sqrt{32}} = 0 \\ b = \frac{5(1-e^{i\beta})}{4\sqrt{32}} \\ c = \frac{5(1-e^{i\beta})}{8\sqrt{32}} \end{cases}.$$

(3) The successful probability

$|4\rangle,|6\rangle,|8\rangle,|10\rangle,|12\rangle,|14\rangle$: $P_1 = 6|b|^2 = \frac{3}{4}$,

$|16\rangle,|18\rangle,|20\rangle,|22\rangle,|24\rangle,|26\rangle,|28\rangle,|30\rangle$: $P_2 = 8|c|^2 = \frac{1}{4}$,

For all marked states: $P = P_1 + P_2 = 1$.

(II) Suppose the quantum superposition state of all marked states is constructed as follows

$$|q\rangle = \sqrt{\frac{25}{32}}|3\rangle + \sqrt{\frac{4}{32}}|9\rangle + \sqrt{\frac{1}{32}}(|15\rangle + |21\rangle + |27\rangle),$$

$$\lambda = \langle q|\phi\rangle^2 = \left(\frac{1}{\sqrt{32}}\left(\sqrt{\frac{25}{32}} + \sqrt{\frac{4}{32}} + 3\sqrt{\frac{1}{32}}\right)\right)^2 = \frac{25}{256},$$

$$(3-\sqrt{5})/8 < \lambda < 1/4,$$

$$\alpha = -\beta = \arccos(1 - (3-\sqrt{5})/(4\lambda)) = \arccos((64\sqrt{5} - 167)/25).$$

According to **theorem 3**, the probability of getting correct results is equal to 1 only two Grover iteration. The search process can be described as follows.

(1) $|\hat{\phi}_1\rangle = (I - (1-e^{i\alpha})|q\rangle\langle q|)|\phi\rangle$

$$= \frac{1}{32}\sum_{j0=1}^{26}|q_{j0}\rangle + \left(\frac{1}{\sqrt{32}} - \frac{5}{16}\sqrt{\frac{25}{32}}(1-e^{i\alpha})\right)|3\rangle + \left(\frac{1}{\sqrt{32}} - \frac{5}{16}\sqrt{\frac{1}{32}}(1-e^{i\alpha})\right)(|15\rangle + |21\rangle + |27\rangle) + \left(\frac{1}{\sqrt{32}} - \frac{5}{16}\sqrt{\frac{4}{32}}(1-e^{i\alpha})\right)|9\rangle.$$

(2) $|\bar{\phi}_1\rangle = ((1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I)|\hat{\phi}_1\rangle$

$$= \begin{pmatrix} aaab & aaaa & acaa & aaad \\ aaaa & adaa & aaad & aaaa \end{pmatrix},$$

where $\begin{cases} a = \dfrac{\sqrt{5}-1}{2\sqrt{32}} \\ b = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{25}{32}}(1-e^{i\beta}) \\ c = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{4}{32}}(1-e^{i\beta}) \\ d = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{1}{32}}(1-e^{i\beta}) \end{cases}.$

(3) $\langle q|\bar{\phi}_1\rangle = [5(\sqrt{5}-1) + 10(1-e^{i\beta})]/32$,

$|\hat{\phi}_2\rangle = (I - (1-e^{i\alpha})|q\rangle\langle q|)|\bar{\phi}_1\rangle$

$$= \begin{pmatrix} aaab & aaaa & acaa & aaad \\ aaaa & adaa & aaad & aaaa \end{pmatrix},$$

where $\begin{cases} a = \dfrac{\sqrt{5}-1}{2\sqrt{32}} \\ b = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{25}{32}}(1-e^{i\beta}) - \langle q|\bar{\phi}_1\rangle\sqrt{\dfrac{25}{32}}(1-e^{i\alpha}) \\ c = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{4}{32}}(1-e^{i\beta}) - \langle q|\bar{\phi}_1\rangle\sqrt{\dfrac{4}{32}}(1-e^{i\alpha}) \\ d = \dfrac{\sqrt{5}-1}{2\sqrt{32}} + \dfrac{5}{16}\sqrt{\dfrac{1}{32}}(1-e^{i\beta}) - \langle q|\bar{\phi}_1\rangle\sqrt{\dfrac{1}{32}}(1-e^{i\alpha}) \end{cases}.$

(4) $|\bar{\phi}_2\rangle = ((1-e^{i\beta})|\phi\rangle\langle\phi| + e^{i\beta}I)|\hat{\phi}_2\rangle$

$$= \begin{pmatrix} aaab & aaaa & acaa & aaad \\ aaaa & adaa & aaad & aaaa \end{pmatrix},$$

where $\begin{cases} a = \dfrac{25(1-e^{i\beta})^2}{256\sqrt{32}} - \dfrac{(\sqrt{5}-3)e^{i\beta}}{2\sqrt{32}} = 0 \\ b = \dfrac{5}{16}\sqrt{\dfrac{25}{32}}(e^{i\beta} - e^{i2\beta}) + \sqrt{\dfrac{25}{32}}\langle q|\bar{\phi}_1\rangle(1-e^{i\beta}) \\ c = \dfrac{5}{16}\sqrt{\dfrac{4}{32}}(e^{i\beta} - e^{i2\beta}) + \sqrt{\dfrac{4}{32}}\langle q|\bar{\phi}_1\rangle(1-e^{i\beta}) \\ d = \dfrac{5}{16}\sqrt{\dfrac{1}{32}}(e^{i\beta} - e^{i2\beta}) + \sqrt{\dfrac{1}{32}}\langle q|\bar{\phi}_1\rangle(1-e^{i\beta}) \end{cases}.$

(5) The successful probability

$|3\rangle$: $P_1 = |b|^2 = 25/32$,

$|9\rangle$: $P_2 = |c|^2 = 1/8$,

$|15\rangle, |21\rangle, |27\rangle$: $P_3 = 3|d|^2 = 3/32$.

For all marked states: $P = P_1 + P_2 + P_3 = 1$.

V. CONCLUSIONS

An improved Grover's algorithm with weighted targets and an adaptive phase matching are proposed. Using this algorithm, the marked states can be searched in an unsorted quantum database, and the probability of each target is coincident with its weight coefficient or importance, which the flexibility of algorithm is shown. With application of the new phase matching, when the inner-product of the target quantum superposition and the initial state of system is greater than $((3-\sqrt{5})/8)^{1/2}$, the probability of getting correct results is equal to 1 after at most two Grover iterations. The validity of the new phase matching is verified by two search examples. Further investigation of this algorithm is needed to increase its success probability when the inner-product mentioned above is less than $((3-\sqrt{5})/8)^{1/2}$.

REFERENCES

[1] L.K. Grover, "A fast quantum mechanical algorithm for database search," Proceedings of the 28th Annual ACM Symposium on the Theory of Computing. 1996, pp.212-221.

[2] P.W. Shor, "Algorithms for quantum computation: Discrete logarithms and factoring," Proceedings of the 35th Annual Symp. on Foundations of Computer Science. 1994, pp.124-134.

[3] L.K. Grover, "Quantum Computers Can Search Rapidly by Using Any Transformation," Physical Review Letters, 1998, pp.4329-4332.

[4] G.L. Long, Y.S. Li, W.L. Zhang, "Phase matching in quantum searching," Physics Letters A, 1999, 262, pp.27-34.

[5] E. Biham, O. Biham, D. Birom, "Grover's quantum search algorithm for an arbitrary initial amplitude distribution," Physical Review A, Vol.60, No.4, 1999, pp.2742-2745.

[6] E. Biham, D. Kenigsberg, "Grover's quantum search algorithm for an arbitrary initial mixed state," Physical Review A, 2002, 66(062301), pp.1-4.

[7] D. Ventura, T. Martinez, "Quantum associative memory," Information Sciences, Vol.124, No.1-4, 2000,pp.273-296.

[8] A. Ezhov, A. Nifanova, D. Ventura, "Quantum associative memory with distributed queries," Information Sciences, Vol.128, No.3-4, 2000, pp.271-293.