On the Feasible Sets of Products of bi-hypergraphs

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Abstract—**In this paper, we discuss respectively the relationships between the feasible sets of the Weak product, the Cartesian product and the disjunctive product of uniform bi-hypergraphs and the feasible sets of the factors.**

Keywords- hypergraph coloring; mixed hypergraph; feasible set; one-realization.

I. INTRODUCTION

A *mixed hypergraph* on a finite set X is a triple $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ families of subsets of X, called the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. A *bi-edge* is an edge which is both a $\mathcal{C}$-edge and a $\mathcal{D}$-edge. If $\mathcal{C}$=$\mathcal{D}$, then $\mathcal{H}$ is called a *bi-hypergraph*. If each edge has r vertices, $\mathcal{H}$ is r-*uniform* if each edge has r vertices. A sub-hypergraph $\mathcal{H}'=(X',\mathcal{C}',\mathcal{D})$ of $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$ is a *spanning sub-hypergraph* if $X'=X$, and is a *derived sub-hypergraph* of $\mathcal{H}$ on X', denoted by $\mathcal{H}[X']$, if $\mathcal{C}'=\{C\in\mathcal{C}|C\subseteq X'\}$ and $\mathcal{D}'=\{D\in\mathcal{D}|D\subseteq X'\}$.

Two mixed hypergraphs $\mathcal{H}_1=(X_1,\mathcal{C}_1,\mathcal{D}_1)$ and $\mathcal{H}_2=(X_2,\mathcal{C}_2,\mathcal{D}_2)$ are *isomorphic* if there exists a bijection ϕ from X_1 to X_2 that maps each $\mathcal{C}$-edge of $\mathcal{C}_1$ onto a $\mathcal{C}$-edge of $\mathcal{C}_2$ and maps each $\mathcal{D}$-edge of $\mathcal{D}_1$ onto a $\mathcal{D}$-edge of $\mathcal{D}_2$, and vice versa. The bijection ϕ is called an *isomorphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$.

A *proper* k*-coloring* of $\mathcal{H}$ is a mapping from X into a set of k *colors* so that each $\mathcal{C}$-edge has two vertices with a *Common* color and each $\mathcal{D}$-edge has two vertices with *Distinct* colors. A *strict* k*-coloring* is a proper k-coloring using all of the k colors, and a mixed hypergraph is k*-colorable* if it has a strict k-coloring. A coloring of $\mathcal{H}$ may be viewed as a partition of its vertex set, where the *color classes* are the sets of vertices assigned to the same color, so a strict n-coloring $c=\{C_1,C_2,\ldots,C_n\}$ of $\mathcal{H}$ means that $C_1,C_2,\ldots,C_n$ are the n color classes under c. The set of all the values k such that $\mathcal{H}$ has a strict k-coloring is called the *feasible set* of $\mathcal{H}$, denoted by $\mathcal{F}(\mathcal{H})$. For each $k\in\mathcal{F}(\mathcal{H})$, let r_k denote the number of *partitions* of the vertex set. For a set S of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a *realization* of S if $\mathcal{F}(\mathcal{H})=S$. A mixed hypergraph $\mathcal{H}$ is a *one-realizatio* of S if it is a realization of S and $r_k=1$ for each $k\in S$.

When one considers the colorings of a mixed hypergraph, it suffices to assume that each $\mathcal{C}$-edge has at least three vertices. The study of the colorings of mixed hypergraphs has made a lot of progress since its inception ([6]). For more information, we would like refer readers to [3, 5, 7, 8].

Perhaps the most intriguing phenomenon of colorings of hypergraphs is that a mixed hypergraph can have gaps in its chromatic spectrum. We know that the feasible set of a classical hypergraph is an interval. Jiang et al.([2]) proved that, for any finite set S of integers greater than 1, there exists a mixed hypergraph $\mathcal{H}$ such that $\mathcal{F}(\mathcal{H})=S$, and Král ([4]) strengthened this result by showing that prescribing any positive integer r_k, there exists a mixed hypergraph which has precisely r_k k-colorings for all $k\in S$. Recently, Bujtás and Tuza ([1]) gave the necessary and sufficient condition for a finite set S of natural numbers being the feasible set of an r-uniform mixed hypergraph. Zhao et al.([9]) proved that any vector $R=(0,r_2,\ldots,r_n)$ with $n\geq 2$ and $r_i\geq 0$, $i=2,\ldots,n$ is the chromatic spectrum of some 3-uniform bi-hypergraph.

In this paper, we focus on the feasible sets of products of uniform bi-hypergraphs with relation to the feasible sets of the factors.

II. MAIN RESULTS

For any positive integer n, let $[n]$ denote the set $\{1,2,\ldots,n\}$.

We focus on the weak product, the Cartesian product and the disjunctive product of uniform bi-hypergraphs, respectively. We first discuss the feasible set of the weak product

Definition 2.1 For any two r-uniform bi-hypergraphs $\mathcal{H}_1=(V_1,\mathcal{B}_1)$ and $\mathcal{H}_2=(V_2,\mathcal{B}_2)$, the weak product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the r-uniform bi-hypergraph $\mathcal{H}_1\times\mathcal{H}_2=(V,\mathcal{B})$, where $V=V_1\times V_2$ and

$$\{(x_1,y_1),\ldots,(x_r,y_r)\}\in\mathcal{B}\Longleftrightarrow$$

$$\{x_1,\ldots,x_r\}\in\mathcal{B}_1, \{y_1,\ldots,y_r\}\in\mathcal{B}_2.$$

Theorem 2.1 Let $\mathcal{H}_1=(V_1,\mathcal{B}_1),\mathcal{H}_2=(V_2,\mathcal{B}_2)$ be two r-uniform bi-hypergraphs. Then

$$\mathcal{F}(\mathcal{H}_1\times\mathcal{H}_2)\supseteq\mathcal{F}(\mathcal{H}_1)\cup\mathcal{F}(\mathcal{H}_2).$$

Proof For any $t\in\mathcal{F}(\mathcal{H}_1)$ and any strict t-coloring $c=\{C_1,C_2,\dots,C_t\}$ of $\mathcal{H}_1$, write

$$C_i'=\{(x,y)\mid x\in C_i, y\in V_2\}, i=1,2,\dots,t,\ \text{and}$$

$$c'=\{C_1',C_2',\dots,C_t'\}.$$

Note that for any $B=\{(x_1,y_1),\dots,(x_r,y_r)\}\in\mathcal{B}$, $B_1=\{x_1,\dots,x_r\}\in\mathcal{B}_1$. Hence, there are two vertices, say x_i,x_j, such that $c(x_i)=c(x_j)$, and two vertices, say x_k,x_m, such that $c(x_k)\neq c(x_m)$. Then

$c'((x_i,y_i)=c'((x_j,y_j))$ and $c'((x_k,y_k))\neq c'((x_m,y_m))$, which implies that c' is a strict t -coloring of $\mathcal{H}_1\times\mathcal{H}_2$. It follows that $\mathcal{F}(\mathcal{H}_1)\subseteq\mathcal{F}(\mathcal{H}_1\times\mathcal{H}_2)$. Similarly, we may have that $\mathcal{F}(\mathcal{H}_2)\subseteq\mathcal{F}(\mathcal{H}_1\times\mathcal{H}_2)$. Thus, the desired result follows.

The following result shows that the equality holds for some uniform bi-hypergraphs. We first construct the desired hypergraphs as follows.

For any set $S=\{n_1,n_2,\dots,n_s\}$ of positive integers with $s\geq 2$ and $\min(S)\geq 2$, let
$X_{n_1,\dots,n_s}=\{(x_1,\dots,x_s)\mid x_j\in[n_j], j\in[s]\}$ and

$$\mathcal{B}_{n_1,\dots,n_s}=\{\{(x_1,\dots,x_s),(y_1,\dots,y_s),(z_1,\dots,z_s)\}\mid . \ |\{x_j,y_j,z_j\}|=2, j\in[s]\}.$$

Then $(X_{n_1,\dots,n_s},\mathcal{B}_{n_1,\dots,n_s})$ is a 3-uniform bi-hypergraph, denoted by $\mathcal{H}_{n_1,\dots,n_s}$. Moreover, we have the following result.

Lemma 2.2 ([9]) Let $S=\{n_1,n_2,\dots,n_s\}$ be a set of positive integers with $s\geq 2$ and $\min(S)\geq 2$. Then

$$\mathcal{F}(\mathcal{H}_{n_1,\dots,n_s})=\{n_1,n_2,\dots,n_s\}.$$

Theorem 2.3 Let

$S_1=\{n_1,\dots,n_s\}, S_2=\{m_1,\dots,n_t\}$ be two sets of integers with $s,t\geq 2$, and $\min(S_1)$, $\min(S_2)\geq 2$. Then there are two 3-uniform bi-hypergraphs, say $\mathcal{H}_1$ and $\mathcal{H}_2$, such that

$\mathcal{F}(\mathcal{H}_1)=S_1,\mathcal{F}(\mathcal{H}_2)=S_2$ and
$\mathcal{F}(\mathcal{H}_1\times\mathcal{H}_2)=S_1\cup S_2$.

Proof By Lemma 2.2, we have that

$$\mathcal{F}(H_{n_1,\dots,n_s})=S_1,\ \mathcal{F}(\mathcal{H}_{m_1,\dots,m_t})=S_2\ \text{and}$$

$$\mathcal{F}(\mathcal{H}_{n_1,\dots,n_s,m_1,\dots,m_t})=S_1\cup S_2.$$

Let

$$\begin{array}{llll}\phi: & X_{n_1,\dots,n_s}\times X_{m_1,\dots m_t} & \to & X_{n_1,\dots,n_s,m_1,\dots,m_t}\\ & ((x_1,\dots,x_s),(y_1,\dots,y_t)) & \to & (x_1,\dots,x_s,y_1,\dots,y_s).\end{array}$$

Then, it is not difficult to notice that ϕ is an isomorphism from $\mathcal{H}_1\times\mathcal{H}_2$ to $\mathcal{H}_{n_1,\dots,n_s,m_1,\dots,m_t}$. Which follows that

$$\mathcal{F}(\mathcal{H}_1\times\mathcal{H}_2)=S_1\cup S_2.$$

Next, we focus on the Cartesian product of bi-hypergraphs.

Definition 2.2 For any two r-uniform bi-hypergraphs $\mathcal{H}_1=(X_1,\mathcal{B}_1),\mathcal{H}_2=(X_2,\mathcal{B}_2)$, the Cartesian product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the r-uniform bi-hypergraph $\mathcal{H}_1\Box\mathcal{H}_2=(X,\mathcal{B})$ with $X=X_1\times X_2$ and

$\{(x_1,y_1),\dots,(x_r,y_r)\}\in\mathcal{B}$ if and only if

$\{x_1,\dots,x_r\}\in\mathcal{B}_1$ and $y_1=\dots=y_r$, or

$\{y_1,\dots,y_r\}\in\mathcal{B}_2$ and $x_1=\dots=x_r$.

Theorem 2.4 Let $\mathcal{H}_1=(X_1,\mathcal{B}_1),\mathcal{H}_2=(X_2,\mathcal{B}_2)$ be two r-uniform bi-hypergraphs. Then

$$\mathcal{F}(\mathcal{H}_1\Box\mathcal{H}_2)\subseteq\mathcal{F}(\mathcal{H}_1)\cap\mathcal{F}(\mathcal{H}_2).$$

Proof For any strict coloring $c=\{C_1,C_2,\dots,C_k\}$ of $\mathcal{H}_1\Box\mathcal{H}_2$ and $y\in X_2$, let

$$\begin{array}{llll}\phi: & X_1 & \to & X_1\times\{y\}\\ & x & \to & (x,y).\end{array}$$

Then ϕ is an isomorphism from $\mathcal{H}_1$ to $(\mathcal{H}_1\Box\mathcal{H}_2)[Y]$, where $Y=X_1\times\{y\}$. Note that is a strict k-coloring of, we have that $\mathcal{H}_1$ is k-colorable. Hence, $\mathcal{F}(\mathcal{H}_1\Box\mathcal{H}_2)\subseteq\mathcal{F}(\mathcal{H}_1)$. Similarly, we may get that $\mathcal{F}(\mathcal{H}_1\Box\mathcal{H}_2)\subseteq\mathcal{F}(\mathcal{H}_2)$, which implies that the desired result follows.

Lastly, we discuss the feasible set of the disjunctive product of bi-hypergraphs.

Definition2.3 For any two r-uniform bi-hypergraphs $\mathcal{H}_1=(X_1,\mathcal{B}_1),\mathcal{H}_2=(X_2,\mathcal{B}_2)$, the disjunctive product of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the r-uniform bi-hypergraph $\mathcal{H}_1*\mathcal{H}_2=(X,\mathcal{B})$, where $X=X_1\times X_2$ and

$\{(x_1,y_1),\dots,(x_r,y_r)\}\in\mathcal{B}$ if and only if

$\{x_1,\dots,x_r\}\in\mathcal{B}_1$, or $\{y_1,\dots,y_r\}\in\mathcal{B}_2$.

Theorem 2.5 Let $\mathcal{H}_1=(X_1,\mathcal{B}_1),\mathcal{H}_2=(X_2,\mathcal{B}_2)$ be two r-uniform bi-hypergraphs. Then

$$\mathcal{F}(\mathcal{H}_1*\mathcal{H}_2)\subseteq\mathcal{F}(\mathcal{H}_1)\cap\mathcal{F}(\mathcal{H}_2).$$

Proof For any strict coloring $c=\{C_1,C_2,\dots,C_k\}$ of $\mathcal{H}_1*\mathcal{H}_2$ and $y\in X_2$, let

$$\begin{array}{llll}\phi: & X_1 & \to & X_1\times\{y\}\\ & x & \to & (x,y).\end{array}$$

Then ϕ is an isomorphism from $\mathcal{H}_1$ to $(\mathcal{H}_1*\mathcal{H}_2)[Y]$, where$Y=X_1\times\{y\}$. Note that

$c' = \{C_1 \cap Y, C_2 \cap Y, \ldots, C_k \cap Y\}$ is a strict k-coloring of $(\mathcal{H}_1 * \mathcal{H}_2)[Y]$, we have $\mathcal{H}_1$ is k-colorable. Thus $k \in \mathcal{F}(\mathcal{H}_1)$ and we further have $\mathcal{F}(\mathcal{H}_1 * \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1)$. Similarly, $\mathcal{F}(\mathcal{H}_1 * \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_2)$. That follows that

$$\mathcal{F}(\mathcal{H}_1 * \mathcal{H}_2) \subseteq \mathcal{F}(\mathcal{H}_1) \cap \mathcal{F}(\mathcal{H}_2).$$

III. CONCLUSIONS

In this paper, we discuss the feasible sets of several products of uniform bi-hypergraphs with relation to the feasible sets of the factors. Precisely, we prove that the feasible set of the Cartesian product or the disjunctive product of two r-uniform bi-hypergraphs is a subset of the intersection of the feasible sets of the factors, and the feasible set of the weak product of two r-uniform bi-hypergraphs contains the union of the feasible sets of the factors.

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