An Algorithm for the Nonlinear Complementarity Problem On Management Equilibrium Model

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Abstract— **In this paper, we consider nonlinear complementarity problem on management equilibrium model (NCP). To solve the problem, we first establish an error bound estimation for the NCP via a new type of residual function. Based on this, the famous Levenberg-Marquardt (L-M) algorithm is employed for obtaining its solution, and we show that the L-M algorithm is quadratically convergent without nondegenerate solution. These conclusions can be viewed as extensions of previously known results.**

Keywords-** **management equilibrium model; nonlinear complementarity problem ; error bound; algorithm

I. Introduction

We consider nonlinear complementarity problem on management equilibrium model (NCP). Let mapping $f: R^n \to R^n$, NCP is to find a vector $x^* \in R^n$ such that

$$x^* \geq 0, \quad f(x^*) \geq 0, \quad (x^*)^T f(x^*) = 0, \qquad (1)$$

where $f(x)$ is polynomial function with power m. We denote the solution set of the NCP by X^*, which is assumed to be nonempty throughout this paper.

The NCP plays a significant role in supply chain equilibrium management, economics management, and operation research, etc. ([1-4]). For example, the balance of supply and demand is central to all economic systems; mathematically, this fundamental equation in economics is often described by complementarity relation between two sets of decision variables. Furthermore, the classical Walrasian law of competitive equilibrium of exchange economies can be formulated as a nonlinear complementarity problem in the price and excess demand variables ([2]).

In recent years, many efficient solution methods have been proposed for solving NCP ([5]). The basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem (e.g., [5, 6, 7, 8]). It is well-known that nonsingularity of Jacobian at a solution guarantees that the famous Levenberg-Marquardt (L-M) method for NCP has a quadratic rate of convergence [6, 7]. Recently, Yamashita and Fukushima showed that the assumption of local error bound is much weaker than that of the nonsingularity of Jacobian [9]. This motivates us to consider the error bound estimation for the NCP. So, in this paper, we are concentrated on establishing an error bound for the NCP via a new type of residual function under mild conditions, and discussing its applications on the convergence analysis of L-M method for solving the NCP.

The rest of this paper is organized as follows. In Section 2, we mainly give equivalent reformulation of NCP, and establish an error bound estimation for the NCP via an easily computable residual function. In Section 3, L-M algorithm is employed for obtaining solution of NCP, and use the obtained result of error bound to establish a quadratic rate of convergence without nondegenerate solution. Compared with the algorithm converges in [6, 7], our conditions are weaker.

We end this section with some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean 2-norm of vector in the space is denoted by $\|\cdot\|$. We also use $x \geq 0$ to denote a nonnegative vector $x \in R^n$ if there is no confusion.

II. The Equivalent Reformulation and Error Bound For NCP

In this section, we present an equivalent reformulation of NCP, and establish an error bound estimation for the NCP, which is extension of previously known result.

First, we give the following Fischer function ([10]) $\phi: R^2 \to R^1$ defined by

$$\phi(a,b) = \sqrt{a^2 + b^2} - a - b, \text{ for } a, b \in R.$$

For this function, besides the following basic property

$$\phi(a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0.$$

For arbitrary vectors $a, b \in R^n$, we also define a vector-valued function $\psi(a,b)$ with

$$\psi(a,b) = (\phi(a_1,b_1), \phi(a_2,b_2), \ldots, \phi(a_n,b_n))^T.$$

Based on this mapping, we can transform NCP into a system of equations via the following vector-valued function $\Phi : R^n \to R^n$ as follows:

$$\Phi(x) := \psi(x, f(x)). \qquad (2)$$

Obviously, the following result is straightforward.

Theorem1. x^* is a solution of the NCP if and only if $\Phi(x^*) = 0$.

To establish an error bound estimation for the NCP by residual function $\Phi(x)$. First, we give the following result in which Tseng ([11]) showed.

Lemma1. For any $(a,b) \in R^2$, one has

$$(2-\sqrt{2})\,|\min(a,b)| \le |\phi(a,b)| \le (2+\sqrt{2})\,|\min(a,b)|. \qquad (3)$$

Lemma2. Given constant $\sigma_0 > 0$, for any $x \in R^n, \|x\| \le \sigma_0$. Then, there exists a constant $\eta_1 > 0$ such that

$$dist(x, X^*) \le \eta_1 \Phi(x)^{\frac{1}{m}}. \qquad (4)$$

where $dist(x, X^*)$ denotes the distance between the point $x \in R^n$ and the solution set X^*.

Proof Firstly, we show that there exists constant $\eta_0 > 0$ such that

$$dist(x, X^*) \le \eta_0 r(x)^{\frac{1}{m}}, \forall x \in R^n, \qquad (5)$$

where $r(x) = \|\min\{x, f(x)\}\|$, $\|x\| \le \sigma_0$.

Assume that the theorem is false. Then, there exist positive sequence $\{\tau_k\}$ and sequence $\{x^k\}$ such that $\tau_k \to \infty$ as $k \to \infty$ and $dist(x^k, X^*) > \tau_k r(x^k)^{\frac{1}{m}}$, where $\|x^k\| \le \sigma_0$. Thus,

$$\frac{r(x^k)^{\frac{1}{m}}}{dist(x^k, X^*)} > \frac{1}{\tau_k} \to 0 \text{ as } k \to \infty. \qquad (6)$$

Since $\{x^k\}$ is bounded and $r(x)$ is continuous, by (6), we have $r(x^k) \to 0$ as $k \to \infty$. Since $\{x^k\}$ is bounded again, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{k_i \to \infty} x^{k_i} = \bar{x}$ with $r(\bar{x}) = 0$. Hence, $\bar{x} \in X^*$. By (6), we can also obtain

$$\lim_{k_i \to \infty} \frac{r(x^{k_i})^{\frac{1}{m}}}{\| x^{k_i} - \bar{x}\|} \le \lim_{k_i \to \infty} \frac{r(x^{k_i})^{\frac{1}{m}}}{dist(x^{k_i}, X^*)} = 0. \qquad (7)$$

On the other hand, since $f(x)$ is polynomial function with power m. From $x^{k_i} \to \bar{x}$, and $r(x^{k_i}) \to 0$. Thus, we know that, for all sufficiently large k_i,

$$r(x^{k_i})^{\frac{1}{m}} / \| x^{k_i} - \bar{x}\| \ge \theta,$$

for some positive number θ, this contradicts (7). Thus, (5) holds.

Secondly, combining (5) with Lemma 1, we have

$$dist(x, X^*) \le \eta_0 r(x)^{\frac{1}{m}} \le \frac{\eta_0 (2+\sqrt{2})}{2} \Phi(x)^{\frac{1}{m}}.$$

Letting $\eta_1 = \frac{\eta_0 (2+\sqrt{2})}{2}$, and the desired result follows.

In this following, we define the function $\rho : R_+ \to R$,

$$\rho(t) = \begin{cases} \bar{\rho}, & \text{if } t \ge \bar{t}, \\ -1/\log t, & \text{if } t \in (0, \bar{t}), \\ 0, & \text{if } t = 0, \end{cases}$$

where $\bar{t} \in (0,1)$ and $\bar{\rho} > 0$ are fixed numbers (the choice of $\bar{\rho}$ does not affect the following theoretical analysis and numerical experiments, and the function $\rho(t)$ is bounded). Combining this with Lemma 2, we have the following result.

Theorem2. Given constant $\sigma_0 > 0$, for any $x \in R^n, \|x\| \le \sigma_0$, there exists a constant $\eta_2 > 0$ such that

$$dist(x, X^*) \le \eta_2 \; \|\rho(\Phi(x))\|.$$

Proof For any given $\nu > 0$, for $\varepsilon_0 > 0$, by

$$\lim_{t \to 0^+} t^{\nu} \log t = 0,$$

we obtain that there exists $\bar{t} > 0$ such that

$$t^{\nu} \le -\varepsilon_0 / (\lg t) = \varepsilon_0 \rho(t), \qquad (8)$$

where $t \in (0, \bar{t})$. Thus, one has

$$\begin{aligned}
dist(x, X^*) &\le \eta_1 \| \Phi(x) \|^{\frac{1}{m}} \\
&= \eta_1 [(\phi(x_1, f_1(x)))^2 + (\phi(x_2, f_2(x)))^2 \\
&\quad + \cdots + (\phi(x_n, f_n(x)))^2]^{\frac{1}{2m}} \\
&\le \eta_1 \{ [(\phi(x_1, f_1(x)))^2]^{\frac{1}{m}} + [(\phi(x_2, f_2(x)))^2]^{\frac{1}{m}} \\
&\quad + \cdots + [(\phi(x_n, f_n(x)))^2]^{\frac{1}{m}} \}^{\frac{1}{2}} \\
&= \eta_1 \left\{ \left\{ [(\phi(x_1, f_1(x)))^2]^{\frac{1}{2m}} \right\}^2 + \left\{ [(\phi(x_2, f_2(x)))^2]^{\frac{1}{2m}} \right\}^2 \right. \\
&\quad \left. + \cdots + \left\{ [(\phi(x_n, f_n(x)))^2]^{\frac{1}{2m}} \right\}^2 \right\}^{\frac{1}{2}} \\
&\le \eta_1 \varepsilon_0^2 \left\{ \rho((\phi(x_1, f_1(x)))^2)^2 + \rho((\phi(x_2, f_2(x)))^2)^2 \right. \\
&\quad \left. + \cdots + \rho((\phi(x_n, f_n(x)))^2)^2 \right\}^{\frac{1}{2}} \\
&= \eta_1 \varepsilon_0^2 \| \rho(\Phi(x)) \|.
\end{aligned}$$

where the first inequality is true by Lemma 2, the fourth inequality is true by (8) with

$$t = [\phi(x_i, f_i(x))]^2, i = 1, 2, \cdots, n,$$

$$v = 1/(2m),$$

the last equality follows from the fact that

$$\rho(\Phi(x))$$

$$= (\rho((\phi(x_1, f_1(x))^2), \rho(\phi(x_2, f_2(x))^2), \cdots, \rho(\phi(x_n, f_n(x))^2))^T,$$

and letting $\eta_2 = \eta_1 \varepsilon_0^2$, then the desired result follows.

In this following, we can transform NCP into a system of equations via the following vector-valued function $\Psi : R^n \to R^n$, and a real-valued function $g : R^n \to R$ as follows:

$$\Psi(x) = \rho(\Phi(x)) \qquad (9)$$

$$g(x) := \frac{1}{2}\Psi(x)^T \Psi(x) = \frac{1}{2}\| \Psi(x)\|^2. \qquad (10)$$

Obviously, we have that the following result hold.

Theorem3. x^* is a solution of the NCP if and only if $\Psi(x^*) = 0$.

III. Algorithm and Convergence

In this section, a Levenberg-Marquardt method for solving the NCP also has quadratic rate of convergence based on the error bound results obtained in Theorem 2. First, we review some definitions and basic results which will be used in the sequel.

In the following, for a locally Lipschitzian mapping $\Theta : R^n \to R^m$, we let $\partial\Theta(x)$ denote the Clarke's generalized Jacobian of $\Theta(x)$ at $x \in R^n$ which can be expressed as the convex hull of the set $\partial_B \Theta(x)$ ([12]), where

$$\partial_B \Theta(x) = \left\{ V \in R^{m\times n} \left| \begin{array}{l} V = \lim_{x^k \to x} \Theta'(x^k), \\ \Theta(x) \text{ is differentiable at } x^k \text{ for all } k \end{array} \right. \right\}.$$

The function $\Phi(x)$ is not differentiable everywhere with respect to $x \in R^n$. However, it is locally Lipschitzian, and therefore has a nonempty generalized Jacobian in the sense of Clarke ([13]). In particular, we present an overestimate of Clarke's generalized Jacobian of $\Phi(x)$. For simplicity, we denote the Clarke's generalized Jacobian of $\Phi(x)$ with respect to $x \in R^n$ by $\partial\Phi(x)$. Similar to the discussion of Proposition 3.1 in [14], we have the following result.

Lemma3. For any $x \in R^n$, we have $\partial\Phi(x) \subseteq (D_a + D_b f'(x))$ where $D_a = \text{diag}(a)$ denotes the diagonal matrix in which the i-th diagonal element is a_i, for vector $a \in R^n$,

$$a_i = \frac{x_i}{\sqrt{x_i^2 + f(x)_i^2}} - 1, b_i = \frac{y_i}{\sqrt{x_i^2 + f(x)_i^2}} - 1, \text{ if } x_i^2 + f(x)_i^2 > 0,$$

$$a_i = \xi_i - 1, \qquad b_i = \eta_i - 1$$

$$\forall\ (\xi_i, \eta_i) \in R^2 \text{ such that } \|(\xi_i, \eta_i)\| \le 1, \text{if } x_i^2 + f(x)_i^2 = 0.$$

Now, we recall some basic definitions about semismoothness and strong semismoothness.

A locally Lipschitz continuous vector valued function $\Theta : R^n \to R^m$ is said to be semismooth at $x \in R^n$, if the limit

$$\lim_{\substack{V \in \partial\Theta(x+th') \\ h' \to h, t\downarrow 0}} \{Vh'\}$$

exists for any $h \in R^n$.

It is well known that the directional derivative, denoted by $\Theta'(x;h)$, of Θ at x in the direction h exists for any $h \in R^n$ if Θ is semismooth at x. The following properties about the semismooth function are due to Qi and Sun in [15].

Lemma4. Suppose that $\Theta : R^n \to R^m$ is a locally Lipschitz function and semismooth, then

a) for any $V \in \partial\Theta(x+h), h \to 0,\ Vh - \Theta'(x;h) = o(\|h\|)$;

b) for any $h \to 0$, $\Theta(x+h) - \Theta(x) - \Theta'(x;h) = o(\|h\|)$.

Semismooth functions lie between Lipschitz functions and continuously differentiable functions, and both continuously differentiable functions and convex functions are semismooth. A stronger notion than semismoothness is strong semismoothness.

The function $\Theta : R^n \to R^m$ is said to be strongly semismooth at x if Θ is semismooth at x and for any $V \in \partial\Theta(x+h)$, $h \to 0$, it holds that

$$Vh - \Theta'(x;h) = O(\|h\|^2).$$

A favorable property of the function $g(x)$ is that it is continuously differentiable on the whole space R^n although $\Psi(x)$ is not in general. We summarize the differential properties of Ψ and g defined by (9) and (10) in the following lemma ([16, 17]).

Lemma5. For the vector-valued function Ψ and real-valued function g defined by (9) and (10), the following statements hold.

(a) Ψ is strongly semi-smooth.

(b) g is continuously differentiable, and its gradient at a point $x \in R^n$ is given by $\nabla g(x) = V^{\bullet}\Psi(x)$, where V is an arbitrary element belonging to $\partial\Psi(x)$.

From Lemma 5 and discussion above, we can obtain the following result.

Theorem4. (a) For some $x^* \in X^*$, there exist constants $\delta \in (0,1)$ and $\eta_3 > 0$ such that

$$\| \Psi(x+h) - \Psi(x) - Vh\| \le \eta_3 \| h\|^2, \forall x+h, x \in \{x \mid \| x - x^*\| \le \delta\}.$$

(b) Ψ is locally Lipschitzian

In this following, a Levenberg-Marquardt method for solving the NCP is outlined. It is similar to that in [9, 6], we consider L-M for NCP with Armijo step size rule, and discuss its global convergence and quadratic rate of convergence.

Algorithm 1

Step 1: Choose any point $x^0 \in R^n$, parameters $\varepsilon \geq 0$, and $\sigma, \beta, \gamma \in (0,1)$. Let $k = 0$.

Step 2: If $\| \nabla g(x^k) \| \leq \varepsilon$, stop; Otherwise, go to Step 3.

Step 3: Choose an element $V^k \in \partial \Psi(x^k)$. Let $d^k \in R^n$ be the solution of the linear system

$$((V^k)^\top V^k + \mu^k I)d = -(V^k)^\top \Psi(x^k).$$

If d^k satisfies $\| \Psi(x^k + d^k) \| \leq \gamma \| \Psi(x^k) \|$, then

$$x^{k+1} := x^k + d^k, k := k+1,$$

go to Step 5. Otherwise, go to Step 4.

Step 4: Let m_k be the smallest non-negative integer m such that

$$g(x^k + \sigma^m d^k) \leq g(x^k) + \beta \sigma^m \nabla g(x^k)^\top d^k.$$

Let $x^{k+1} := x^k + \sigma^{m_k} d^k$.

Step 5: Set $\mu^{k+1} = \| \Psi(x^{k+1}) \|^2, k := k+1,$ go to Step 2.

For the above Algorithm 1, we assume that Algorithm 1 generates an infinite sequence $\{x^k\}$. Using Theorem 2 and 4, combining the proof of Theorem 3.1 in [9], we can obtain the following global convergence Theorem.

Theorem5. Let $\{x^k\}$ be generated by Algorithm 1, then any accumulation point of the sequence $\{x^k\}$ is a stationary point of g. Moreover, if an accumulation point x^* of the sequence $\{x^k\}$ is a solution of (9). Then $dist(x^k, X^*)$ converges to 0 quadratically.

In Theorem 5, the L-M algorithm is employed for obtaining solution of NCP, and the method has quadratic rate of convergence under local error bound, which is much weaker than the nonsingularity of Jacobian. It is an extension of the algorithm converges conclusion in [6, 7], which is a new result for NCP.

In the end of this section, we present the following example on the supply chain network equilibrium problem [18], and the numerical experiment of the example is also reported. Now, we will implement Algorithm 1 in Matlab and run it on a Pentium IV computer. We take parameter $\varepsilon = 10^{-16}$, **Iter** denotes the number of iterations.

Example We let $f(x) = Mx + p$, where

$$M = \begin{pmatrix} 6 & 1 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 2 & 0 & 2 & 2 \\ -1 & -1 & -2 & 0 \end{pmatrix}, p = \begin{pmatrix} -9 \\ -6 \\ -4 \\ 3 \end{pmatrix}$$

Using Algorithm 1, we obtain the solution of this problem

$$x^* = (\frac{4}{3}, \frac{7}{9}, \frac{4}{9}, \frac{2}{9})^T.$$

The results for this problem with different starting points are shown in Table 1. We see that the Algorithm 1 performs well for this problem. To illustrate the stability of Algorithm 1, the initial point x^0 is produced randomly in (0, 1.5), we use it to solve example, and the results are listed in Table 2. Table 1 and Table 2 indicate that Algorithm 1 is not sensitive to the change of initial point, thus it is very stable.

Table 1. Numerical Results of this Example

Starting point	$(0,0,0)^T$	$(1,1,1)^T$	$(-1,-1,-1)^T$
Iter	3	4	8

Table 2. Numerical Results of this Example

Trial	1	2	3	4
Iter	2	5	2	3
Trial	5	6	7	8
Iter	6	4	2	3

IV. CONCLUSIONS

In this paper, we first established error estimation for nonlinear complementarity problem on management equilibrium model (NCP) by a new residual function. Then, we propose an algorithm to solve the NCP, and use the error estimation to establish the global and quadratic rate of convergence without nondegenerate solution instead of the nonsingular assumption, this conclusion can be viewed as extension of previously known result ([6, 7]). How to use our algorithm to solve the practical management problem based on the computer, this is a topic for future research.

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