An Error Estimation for Management Equilibrium Model

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Abstract—**In this paper, we extended the result about the error estimation for linear complementarity problem to the extended linear complementarity problem (ELCP) in management equilibrium modeling. More precisely, we first present that the level set of a computable residual function is bounded, and give error estimation in level set. Based on this, we use this residual function to establish a global error estimation of the ELCP. The result obtained in this paper can be viewed as extensions of previously known results.**

Keywords- management equilibrium model; error estimation; extended linear complementarity problem

I. Introduction

Let mappings $F(x)=Mx+p,\quad G(x)=Nx+q$,the extended linear complementarity problem, abbreviated as ELCP, is to find vector $x^*\in R^n$ such that

$$\begin{array}{c} F(x^*)\geq 0,\quad G(x^*)\geq 0,\quad F(x^*)^{\bullet}G(x^*)=0,\\ Ax^*+a\leq 0,\quad Bx^*+b=0, \end{array} \tag{1}$$

where $M,N\in R^{m\times n},p,q\in R^m,A\in R^{s\times n},B\in R^{t\times n},a\in R^s,b\in R^t$. We assume that the solution set of the ELCP is nonempty throughout this paper, and denote it by X^*.

The ELCP is a direct generalization of the classical linear complementarity problem (LCP) which finds applications in management equilibrium, economics, finance, and operation research (Refs.[1,2]). For example, the coupling effect between technology innovation and management innovation is the essence of the business model innovation; the balance of supply and demand is central to all economic systems. Mathematically, this fundamental equation in economics is often described by a complementarity relation between two sets of decision variables. Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables ([2]).

Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature ([3]). Among all the useful tools for theoretical and numerical treatment to complementarity problems, the global error estimation is an important one. The error estimation for the classical linear complementarity problems (LCP) was fully analyzed ([4-9]). For example, Mangasarian and Shiau ([5]) are the first one who gave the error bound analysis to linear complementarity problems. Latter, Mathias and Pang ([6]) established the global error bound estimation for the LCP with a P-matrix in terms of the natural residual function, and Mangasarian and Ren gave the same error bound of the LCP with an R_0-matrix in [7]. Since the ELCP is an extension of the LCP, and this motivates us to consider the following two questions naturally: How about the error bound estimation for the ELCP? Can the existing error estimation for the LCP be extended to the ELCP? These constitute the main topics of this paper.

The main contribution of this paper is to establish a global error estimation for the ELCP via an easily computable residual function, which can be taken as an extension of those for linear complementarity problems ([7,8,9]).

We end this section with some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean norm of vector in the space is denoted by $\|\cdot\|$. We use R^n_+ to denote the nonnegative orthant in R^n , use x_+ and x_- to denote the vectors composed by elements $(x_+)_i:=\max\{x_i,0\}$, $(x_-)_i:=\max\{-x_i,0\},1\leq i\leq n$, respectively. We also use $x\geq 0$ to denote a nonnegative vector $x\in R^n$ if there is no confusion.

II. The Error Estimation For ELCP

In this section, we present error estimation for ELCP, which is extension of previously known results. First, we give the following definition.

Definition1. The matrix Q is said to be R_0-matrix if the following systems has only a zero solution

$$x\geq 0,Qx\geq 0,x^TQx=0.$$

To proceed, we present the following some lemmas which will be needed in the sequel.

Lemma1. Suppose that M^TN is R_0 -matrix, for any sequence $\{x^k\}\subseteq R^n$ satisfying $\|x^k\|\to\infty$,

$$\liminf_{k\to\infty}\min_{1\le i\le m}\frac{F_i(x^k)}{\|x^k\|}\ge 0, \quad (2)$$

$$\liminf_{k\to\infty}\min_{1\le i\le m}\frac{G_i(x^k)}{\|x^k\|}\ge 0, \quad (3)$$

then there is an index j such that

$$\liminf_{k\to\infty}\{F_j(x^k)\}/\|x^k\|>0,$$
$$\liminf_{k\to\infty}\{G_j(x^k)\}/\|x^k\|>0.$$

Proof Suppose that $\{x^k\}$ be unbounded sequence. Let $y^k=\frac{x^k}{\|x^k\|}$, and $\{y^{k_i}\}$ be a subsequence of $\{y^k\}$ and converges to $\bar{y}\neq 0$, we can obtain

$$M\bar{y}=\lim_{k_i\to\infty}(Mx^{k_i}+p)/\|x^{k_i}\|$$
$$\ge\liminf_{k_i\to\infty}\min_{1\le j\le m}\{\{F_j(x^{k_i})\}/\|x^{k_i}\|\}e$$
$$\ge 0,$$
$$N\bar{y}=\lim_{k_i\to\infty}(Nx^{k_i}+q)/\|x^{k_i}\|$$
$$\ge\liminf_{k_i\to\infty}\min_{1\le j\le m}\{\{G_j(x^{k_i})\}/\|x^{k_i}\|\}e$$
$$\ge 0,$$

where $e=(1,1,\cdots,1)^T$. Since M^TN is R_0 -matrix, then $\bar{y}^TM^TN\bar{y}>0$. Thus, the desired result follows.

Lemma2. Suppose that M^TN is R_0 -matrix, for any $\varepsilon>0$, let

$$X_\varepsilon=\{x\in\Omega\,|\,r(x)\le\varepsilon\},$$

then X_ε is bounded, where

$$r(x)=\|\min\{F(x),G(x)\}\|,$$
$$\Omega=\{x\in R^n\,|\,Ax+a\le 0, Bx+b=0\}.$$

Proof: Suppose that X_ε is unbounded for some $\varepsilon>0$, then there exists sequence $\{x^k\}\in X_\varepsilon$ such that

$$\|x^k\|\to\infty \text{ as } k\to\infty.$$

If the sequence $\{x^k\}$ satisfy (2) and (3), combining Lemma 1 with Definition 1, then there is an index j such that

$$\liminf_{k\to\infty}\left\{\frac{F_j(x^k)}{\|x^k\|}\right\}>0,$$
$$\liminf_{k\to\infty}\left\{\frac{G_j(x^k)}{\|x^k\|}\right\}>0.$$

Thus, we can obtain $F_j(x^k)\to\infty$, $G_j(x^k)\to\infty$ as $k\to\infty$. i.e.,

$$r(x^k)=\|\min\{F(x^k),G(x^k)\}\|$$
$$\ge\sqrt{\min\{F_j(x^k),G_j(x^k)\}^2}.$$
$$\to\infty \quad (k\to\infty).$$

This leads to a contradiction.

On the other hand, if the sequence $\{x^k\}$ does not satisfy (2) or (3), then there are subsequence K and indices i and j such that

$$\text{either} \liminf_{k\to\infty}\{F_i(x^k)/\|x^k\|\}<0$$
$$\text{or} \liminf_{k\to\infty}\{G_j(x^k)/\|x^k\|\}<0, k\in K.$$

Thus, we have

$$\text{either} \lim_{k\to\infty}F_i(x^k)=-\infty \quad\text{or}\quad \lim_{k\to\infty}G_j(x^k)=-\infty, k\in K.$$

i.e.,

$$r(x^k)=\|\min\{F(x^k),G(x^k)\}\|$$
$$\ge(F_i(x^k)^2+G_j(x^k)^2)^{\frac{1}{2}}$$
$$\to\infty \quad (k\to\infty, k\in K),$$

which also leads to a contradiction.

Lemma3. Suppose that M^TN is an R_0 -matrix, and $X^*\neq\emptyset$, then there exists a constant $\eta_1>0$ such that

$$dist(x,X^*)\le\eta_1 r(x), \quad \forall x\in X_\varepsilon. \quad (4)$$

where $dist(x,X^*)$ denotes the distance between the point $x\in R^n$ and the solution set X^*.

Proof Assume that the theorem is false. Then there exist $\varepsilon_0>0$, positive sequence $\{\tau_k\}$ and sequence $\{x^k\}\subseteq X_{\varepsilon_0}$ such that $\tau_k\to\infty$ as $k\to\infty$ and $dist(x^k,X^*)>\tau_k r(x^k)$. Thus,

$$\frac{r(x^k)}{dist(x^k,X^*)}>\tau_k\to 0 \text{ as } k\to\infty. \quad (5)$$

Since X_{ε_0} is bounded and $r(x)$ is continuous, by (5), we have $r(x^k)\to 0$ as $k\to\infty$. Since X_{ε_0} is bounded again and $\{x^k\}\subseteq X_{\varepsilon_0}$, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{k_i\to\infty}x^{k_i}=\bar{x}\in X_{\varepsilon_0}$ with $r(\bar{x})=0$. Hence, $\bar{x}\in X^*$. By (5), we can also obtain

$$\lim_{k_i\to\infty}\frac{r(x^{k_i})}{\|x^{k_i}-\bar{x}\|}\le\lim_{k_i\to\infty}\frac{r(x^{k_i})}{dist(x^{k_i},X^*)}=0. \quad (6)$$

On the other hand, since $F(x)$ and $G(x)$ are both polynomial functions with powers 1, respectively. From $x^{k_i}\to\bar{x}$, and $r(x^{k_i})\to 0$. Thus, we know that, for all sufficiently large k_i,

$$\frac{r(x^{k_i})}{\|x^{k_i}-\bar{x}\|}\ge\theta,$$

for some positive number θ, this contradicts (6). Thus, (4) holds.

Based on Lemma 3 and the globalization technique by Mangasarian and Ren ([9]), we have the following conclusion.

Lemma4. Suppose that $M^T N$ is an R_0 -matrix, and $X^* \neq \emptyset$, then there exist constants $\eta_2 > 0$ such that

$$dist(x, X^*) \leq \eta_2 r(x), \forall x \in \Omega.$$

Proof Assume that the theorem is false. Then for any integer k, there exist $x^k \in \Omega$ and $\bar{x} \in X^*$, such that

$$\| x^k - \bar{x} \| > kr(x^k). \tag{7}$$

It follows that there exist $k_0 > 0, \varepsilon_0 > 0, \forall k > k_0$, we have

$$r(x^k) > \varepsilon_0. \tag{8}$$

In fact, otherwise, for $\forall k > 0, \forall 0 < \varepsilon < 1$, there exists $\bar{k} > k$, such that $r(x^{\bar{k}}) \leq \varepsilon$, we let $X_\varepsilon = \{x \in \Omega \mid r(x) \leq \varepsilon\}$. Combining this with Lemma 2, we know that X_ε is bounded. Using Lemma 3 again, for $x^{\bar{k}} \in X_\varepsilon$, there exist $\bar{x}(x^{\bar{k}}) \in X^*$ and constant $\eta_1 > 0$, such that

$$\| x^{\bar{k}} - \bar{x}(x^{\bar{k}}) \| \leq \eta_1 r(x^{\bar{k}}).$$

Combining this with (7), we have

$$\frac{\eta_1}{\bar{k}} \| x^{\bar{k}} - \bar{x}(x^{\bar{k}}) \| > \eta_1 r(x^{\bar{k}}) \geq \| x^{\bar{k}} - \bar{x}(x^{\bar{k}}) \| .$$

Thus, we have $\frac{\eta_1}{\bar{k}} > 1$. Let $\bar{k} \to \infty$, then we have $\frac{\eta_1}{\bar{k}} < 1$, this is contradiction, we have that (8) holds. Combining (7) with (8), we have

$$\| x^k - \bar{x} \| > kr(x^k) > k\varepsilon_0,$$

i.e., $\| x^k \| > k\varepsilon_0 - \| \bar{x} \|$. Thus, we can obtain

$$\| x^k \| \to \infty (k \to \infty).$$

Since $M^T N$ is an R_0 -matrix, by Lemma 1. We have the following conclusion holds

$$\lim_{k \to \infty} \frac{r(x^k)}{\| x^k \|} > 0. \tag{9}$$

In fact, if the sequence $\{x^k\}$ does not satisfy (2) or (3), then there exist indices i and j such that

$$\text{either } \liminf_{k \to \infty} \left\{ \frac{F_i(x^k)}{\|x^k\|} \right\} < 0 \text{ or } \liminf_{k \to \infty} \left\{ \frac{G_j(x^k)}{\|x^k\|} \right\} < 0.$$

Thus, we have (9) holds. If the sequence $\{x^k\}$ satisfies (2) and (3), from Lemma 1, we also have (9) holds.

Let $y^k = \frac{x^k}{\| x^k \|}$, then there exist a subsequence $\{y^{k_i}\}$ of $\{y^k\}$, such that $y^{k_i} \to \bar{y}$ $(k_i \to \infty)$, note that $\| \bar{y} \| = 1$. Divide both sides of (7) by $\| x^{k_i} \|$, and let k_i go to infinity, and using (9), we obtain

$$1 = \lim_{i \to \infty} \frac{\| x_{k_i} - \bar{x} \|}{\| x^{k_i} \|} > \lim_{i \to \infty} \frac{k_i r(x^{k_i})}{\| x^{k_i} \|} \to \infty,$$

this is contradiction, then the desired result is followed.

In the following, we give the definition of projection operator and some relate properties ([10]). For nonempty closed convex set $\bar{\Omega} \subset R^n$ and any vector $x \in R^n$, the orthogonal projection of x onto $\bar{\Omega}$, i.e.,

$$argmin\{\| y - x \| \mid y \in \bar{\Omega}\},$$

is denoted by $P_{\bar{\Omega}}(x)$.

Lemma5. For any $u, v \in R^n$, then

$$\| P_{\bar{\Omega}}(u) - P_{\bar{\Omega}}(v) \| \leq \| u - v \|.$$

We also give the needed result from Ref. [12] mainly discusses the error bound for a polyhedral cone.

Proposition1. For polyhedral cone

$$P = \left\{ x \in R^n \middle| D_1 x = d_1, D_2 x \leq d_2 \right\}$$

with $D_1 \in R^{l \times n}, D_2 \in R^{m \times n}$, $d_1 \in R^l, d_2 \in R^m$, there exists constant $c > 0$ such that

$$\text{dist}(x, P) \leq c[\| D_1 x - d_1 \| + \| (D_2 x - d_2)_+ \|] \quad \forall x \in R^n.$$

Base on Proposition 1, we are at the position to state our main results in the following.

Theorem1. Suppose that $M^T N$ is an R_0 -matrix, and $X^* \neq \emptyset$. Then there exists constant $\rho > 0$ such that

$$dist(x, X^*) \leq \rho \left[r(x) + \| (Ax + a)_+ \| + \| Bx + b \| \right], \quad \forall x \in R^n.$$

Proof By Proposition1, for given $x \in R^n$, we only need to first project x to Ω, i.e., there exist vector $\bar{x} \in \Omega$ and constant $\tau > 0$ such that

$$\text{dist}(x, \Omega) = \| x - \bar{x} \| \leq \tau \left(\| (Ax + a)_+ \| + \| Bx + b \| \right).$$

Since

$$\begin{aligned}
&\| r(x) - r(\bar{x}) \| \\
&= \| \min\{F(x), G(x)\} - \min\{F(\bar{x}), G(\bar{x})\} \| \\
&= \| [F(x) - P_{R_+}(F(x) - G(x))] \\
&\quad - [F(\bar{x}) - P_{R_+}(F(\bar{x}) - G(\bar{x}))] \| \\
&\leq \| [F(x) - F(\bar{x}) \| \\
&\quad + \| P_{R_+}(F(x) - G(x)) - P_{R_+}(F(\bar{x}) - G(\bar{x})) \| \\
&\leq \| F(x) - F(\bar{x}) \| \\
&\quad + \| (F(x) - G(x)) - (F(\bar{x}) - G(\bar{x})) \| \\
&\leq 2 \| F(x) - F(\bar{x}) \| + \| G(x) - G(\bar{x}) \| \\
&\leq 2 \| M \| \ \| x - \bar{x} \| + \| N \| \ \| x - \bar{x} \| \\
&= (2 \| M \| + \| N \|) \| x - \bar{x} \| \\
&= (2 \| M \| + \| N \|) \ dist(x, \Omega).
\end{aligned}$$

where the second inequality is by Lemma 5, and we have

$$r(\bar{x}) \le r(x) + (2\|M\| + \|N\|)dist(x,\Omega). \qquad (10)$$

Combining (10) with Proposition 1, we have

$$\begin{aligned} dist(x, X^*) &\le dist(x,\Omega) + dist(\bar{x}, X^*) \\ &\le dist(x,\Omega) + \eta_2 r(\bar{x}) \\ &\le dist(x,\Omega) \\ &\quad + \eta_2(r(x) + (2\|M\| + \|N\|)dist(x,\Omega)) \\ &\le \eta_2 r(x) \\ &\quad + [\eta_2(2\|M\| + \|N\|) + 1]dist(x,\Omega) \\ &\le \eta_2 r(x) + [\eta_2(2\|M\| + \|N\|) + 1]\tau \\ &\quad \square(\|(Ax+a)_+\| + \|Bx+b\|) \\ &\le \rho\left[r(x) + \|(Ax+a)_+\| + \|Bx+b\|\right], \end{aligned}$$

where $\rho = \max\{[\eta_2, [\eta_2(2\|M\| + \|N\|) + 1]\tau\}$.

Remark The error bound in the above Theorem 1 is extensions of Theorem 2.6 in [9], Theorem 2.1 in [7], Theorem 4.3 in [8] for linear complementarity problem, and is also extensions Theorem 4.2 in [10].

In the end of this section, we present the following example in enterprise innovation performance management, and the numerical experiment of the error estimation in the above Theorem 1 is also reported in this paper.

Example Enterprise innovation performance depends on innovational elements such as technical factors, culture, organization structure, property right, incentive system, human resource management, etc. Let's say that we have *n* different innovational elements, the quantity demanded and the quantity supplied as functions of the price of the elements p_1, $p_2, ..., p_n$. For comparison, we let $p_1, p_2, ..., p_n$ is relative price such that

$$p = (p_1, p_2, ..., p_n)^T \ge 0,\ p_1 + p_2 + ... + p_n = 1.$$

Supply function $S_i(p) = S_i(p_1, p_2, \cdots, p_n),\quad i = 1, \cdots, n$.
Demand function $D_i(p) = D_i(p_1, p_2, \cdots, p_n),\ i = 1, \cdots, n$.
Quantity supplied is not less than quantity demanded, then

$$S_i(p) \ge D_i(p), i = 1, \cdots, n..$$

According to Walrasian Law, we have

$$\sum_{i=1}^{n} p_i S_i(p) = \sum_{i=1}^{n} p_i D_i(p)..$$

Based on the analysis above, this problem is to find a vector $\overline{p} = (\overline{p_1}, \cdots, \overline{p_n})$, such that

$$\begin{cases} S(\overline{p}) - D(\overline{p}) \ge 0, \overline{p} \ge 0 \\ \left(\overline{p}\right)^T \left(S(\overline{p}) - D(\overline{p})\right) = 0 \\ e^T \overline{p} = 1 \end{cases}$$

where

$$S(\overline{p}) - D(\overline{p}) = (S_1(\overline{p}) - D_2(\overline{p}), \cdots, S_n(\overline{p}) - D_n(\overline{p}))^T,$$

$$e = (1, 1, ..., 1)^T.$$

According to Daft ([16]) and Birkinshaw et al. ([15]), these n different innovational elements could be further reduced to technology and management elements.

Now, we present the following an example:

$$S(p) - D(p) = Mp \ge 0,\quad p = \left(p_1, p_2\right)^T \ge 0,$$

$$\left(p_1, p_2\right) M \left(p_1, p_2\right)^T = 0,\quad p_1 + p_2 = 1,$$

where $M = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$.

Obviously, the solution set of this problem $P^* = \{0\}$, and the matrix M is a R_0 -matrix. For any

$$p(\varepsilon) := (1-\varepsilon, \varepsilon)^T, \varepsilon \ge 0,$$

we have

$$\frac{\|p(\varepsilon) - 0\|}{\|1 - \varepsilon + \varepsilon - 1\| + \|\min\{p(\varepsilon), Mp(\varepsilon)\}\|}.$$

$$= \frac{\sqrt{2\varepsilon^2 - 2\varepsilon + 1}}{\sqrt{17\varepsilon^2 - 8\varepsilon + 1}} \to 1(\text{ò} \to 0).$$

Thus, global error estimation in Theorem1in this paper for this example is provided.

III. Conclusions

In this paper, we established global error estimation on the extended linear complementarity problems in management equilibrium modeling, which are the extensions of those for the classical linear complementarity problems. Surely, we may use the error estimation to establish quick convergence rate of the Newton-type method for solving the ELCP ([13]) instead of the nonsingular assumption just as was done for nonlinear equations ([14]), this is a topic for future research.

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