Periodic Error Correcting Perfect Codes

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Abstract— **In this paper, we present a new class of perfect codes to be known as periodic error correcting perfect codes. Periodic errors were introduced as alternate errors by Tyagi and Das (2010). The occurrence of periodic errors can be seen in communication channels like Astrophotography, Gyroscope and in computed tomography. These errors are caused by run out or gear teeth spacing discrepancies in the mount's RA (Right Ascension) worm gears. It becomes "periodic" in a sense that as the worm gear rotates, you encounter the same problematic part of the gear over and over again. The time period of the error depends on how long the worm gear completes one revolution. If one revolution is completed after s-components, then the errors are termed as s-periodic errors. By a periodic error, we mean a vector whose non zero components are located at a certain fixed shifting positions in a code vector. For example, the 1-periodic error *vectors* are the ones where errors occur in** $1^{st}, 3^{rd}, 5^{th}\ldots$ **positions or** $2^{nd}, 4^{th}, 6^{th}, \ldots$ **Positions and so on. Such codes are studied by Tyagi and Das.**

Keywords- Periodic errors, parity check matrix, syndromes and perfect codes.

I. INTRODUCTION

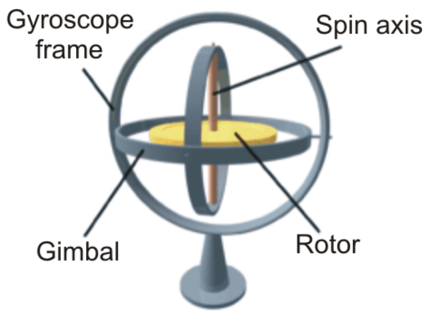
A system's noise environment can create errors in the transmitted message. Properties of the resulting errors depend upon characteristics of the channel and system. Errors which are encountered can be classified mainly into two categories.

(i) **Random errors:** The bit errors are independent of each other. Additive noise typically causes random errors.
(ii) **Burst errors:** The bit errors occur sequentially in time and as groups. Media defects in digital storage systems can cause burst errors.

Most of the studies in error correcting codes are with respect to these two categories of errors. Codes correcting such errors are known as Random error correcting codes and Burst error correcting codes. In certain other communication channels like: Astrophotography [3], Gyroscope [4] and Computed Tomography [5], it has been found that whenever error occurs, it is in periodic form.

1) **Astrophotography —** Where small mechanical error occurs periodically in the accuracy of the tracking in a motorized mount that results small movements of the target that can spoil long-exposure images, even if the mount is perfectly polar-aligned and appears to be tracking perfectly in short tests. It repeats at a regular interval — the interval being the amount of time or period it takes the mount's drive gear to complete one revolution**.**

2) **Gyroscope —** Gyroscope has periodic error items and the gyroscope output need data even process before navigate calculation in strap-down inertial navigation system. The frequency error of period item in zero drift is the main factor to influence the gyroscope accuracy by the analysis method of seeking every period parameter's error transfer coefficients with error analysis theories. The influence of the data smooth process upon the frequency characteristics of period item is described in mathematics form and the frequency transfer model is established, then the frequency transfer characteristics are analyzed.



3) **Computed Tomography —** Structured noise in computed tomography effects of periodic error sources. The artifact in computed tomography (CT) images due to cyclic projection errors, such as errors due to periodic fluctuations in x-ray intensity, is derived and verified by computer simulation. The radius of appearance of this fundamental artifact is independent of the frequency of the periodic error signal and will only be visible in 4-G CT scanners. The effects of sampling are derived and illustrated by simulation for first-, third-, and fourth-generation CT-scanner geometries. The radius of appearance of all but the fundamental artifact is shown to be dependent on the frequency of the periodic signal.

Hence there is a need to develop codes that can take care of periodic errors. Tyagi and Das (2010, 2012) have introduced codes that can detect and correct s-alternate errors. We have named these alternate errors as periodic errors in this paper. The paper is organized into four sections. In section II, we give some basic definitions that are frequently used in this paper. Section III deals with s -periodic error correcting perfect codes (PECPC). Conclusions and open problems are given in section IV.

II. Definitions

s **-periodic error.** A s-periodic error is an n-tuple whose non zero components are located at a gap of s positions and the number of its starting positions is first $s+1$ components, where $s=1,2,3,\ldots,(n-1)$. For $s=1$, *the* 1 *-periodic error vectors* are the ones where error may occur in $1^{st},3^{rd},5^{th}$ positions or $2^{nd},4^{th},6^{th},\ldots$ positions.

For example, in a vector of length 8, 1-*periodic error vectors* are of the type 10101000, 00101000, 10101010, 10001010, 01010101, 01000101, 00000101, etc.

Similarly, for $s=2$, the 2-*periodic error vectors* are those where error may occur in $1^{st},4^{th},7^{th},\ldots$ positions or $2^{nd},5^{th},8^{th},\ldots$ positions or $3^{rd},6^{th},9^{th},\ldots$ positions.

The 2-*periodic error vectors* may look like 10010010, 10000010, 00010010, 01001001, 01000001, 00001001, etc. in a vector of length 8.

Thus, if s -periodic error occurs in an n -tuple $(a_1,a_2,a_3,a_4,\ldots,a_{s-2},a_{s-1},a_s,a_{s+1},\ldots,a_{2s},a_{2s+1},\ldots,a_n)$, then the location of errors can be given in the following $s+1$ exclusive sequences:

$$\{a_1,a_{s+2},a_{2s+3},\ldots\},\{a_2,a_{s+3},a_{2s+4},\ldots\},$$
$$\{a_3,a_{s+4},a_{2s+5},\ldots\},\ldots,\{a_{s-1},a_{2s},a_{3s+1},\ldots\},$$
$$\{a_s,a_{2s+1},a_{3s+2},\ldots\},\{a_{s+1},a_{2s+2},a_{3s+3},\ldots\}.$$

s **-periodic error correcting (** s **-PEC) linear code.** A code is called s-*PEC linear code* if it corrects all s-periodic errors.

III. PERFECT CODES

Tyagi and Das (2010) have obtained necessary and sufficient bounds over the number of parity check digits required for correcting s-periodic errors in a linear code. The lower bound on the necessary number of parity check digits required for an (n,k) linear code over $\mathrm{GF}(q)$ is as follows:

Theorem (Tyagi and Das, 2010). The number of parity check digits in a linear code over $\mathrm{GF}(q)$ that corrects all s-periodic errors is at least

$$\log_q\left[1+\sum_{i=0}^{s}(q^{k_i}-1)\right] \tag{1}$$

Where $k_i=\left\lceil\frac{n-i}{s+1}\right\rceil$; $i=0,1,2,\ldots,s$.

Whenever one obtains a bound, it is desirable to know as to for which values of the parameters the bound is realized. To look into the values of various parameters for which the bound is tight, we must consider inequality in (1) as equality, viz.

$$q^{n-k}=1+\sum_{i=0}^{s}(q^{k_i}-1) \tag{2}$$

For binary case, we have

$$2^{n-k}=1+\sum_{i=0}^{s}(2^{k_i}-1) \tag{3}$$

In order to obtain s-periodic error correcting perfect codes, we now examine the values of k_i, s and $n-k$.

(i) For $s=2$, equality (3) becomes

$$2^{n-k}=2^{k_0}+2^{k_1}+2^{k_2}-2 \tag{4}$$

We now examine the values to n, k_0, k_1, k_2 to check the possibilities of 2-periodic perfect codes.

Case 1. For $n=3$ and $k_0=k_1=k_2=1$, from (4) we obtain the value of $k=1$. This may give us $(3,1)$ 2-periodic perfect code.

Case 2. For $n=5$ and $k_0=k_1=k_2$ and $k_2=1$ the equality (4) gives the value of $k=2$. This may result into $(5,2)$ 2-periodic perfect code.

(ii) For $s=4$, equality (3) becomes

$$2^{n-k}=2^{k_0}+2^{k_1}+2^{k_2}+2^{k_3}+2^{k_4}-4 \tag{5}$$

Here also, we examine the values to n, k_0, k_1, k_2, k_3 and k_4 to check the possibilities of 4-periodic perfect codes.

Case 1. For $n=6$, $k_0=2$, $k_1=k_2=k_3k_4=1$, the equality (5) gives $k=3$. This may give rise to $(6,3)$ 4-periodic perfect code.

Consider the matrix

$$H=\begin{pmatrix}1&0&0&1&1&0\\0&1&0&0&1&1\\0&0&1&1&0&1\end{pmatrix}_{3\times 6}$$

If we consider the matrix H as the parity check matrix for $(6,3)$ periodic perfect code, then it can be verified from the error pattern-syndrome table that the code is a perfect code.

Error Pattern	Syndrome
100000	100
000001	011
100001	111
010000	010
001000	001
000100	101
000010	110

Case 2. For $n=10$, $k_0=k_1=k_2=k_3=k_4=2$, the equality (5) gives $k=6$. This may give rise to $(10,6)$ 4-periodic perfect code.

Let us consider the following matrix as parity check matrix for $(10,6)$ code.

$$H=\begin{pmatrix}1&0&0&0&1&0&0&1&1&0\\0&1&0&0&0&1&0&0&1&1\\0&0&1&0&1&1&1&0&0&0\\0&0&0&1&0&0&1&1&0&1\end{pmatrix}_{4\times 10}$$

It can be verified from the following error pattern and syndrome table that $(10,6)$ code is a 4-*periodic perfect code.*

Error Pattern	Syndrome
1000000000	1000
1000010000	1110
0000010000	0110
0100000000	0100
0100001000	0111
0000001000	0011
0010000000	0010
0010000100	1011
0000000100	1001
0001000000	0001
0001000010	1101
0000000010	1100
0000100000	1010
0000100001	1111
0000000001	0101

This shows that the $(10,6)$ code whose parity check matrix is given above is 4-periodic perfect code.

(iii) For $s=6$, the equality (3) becomes

$$2^{n-k}=2^{k_0}+2^{k_1}+2^{k_2}+2^{k_3}+2^{k_4}+2^{k_5}+2^{k_6}-6 \tag{6}$$

We now examine the values to $n, k_0, k_1, k_2, k_3, k_4, k_5$ and k_6 to check the possibilities of 6-periodic Perfect codes.

*Case*1

For $n=11$, $k_0=k_1=k_2=k_3=2, k_4=k_5=k_6=1$, the equality (6) gives $k=7$. This shows the possibility of the existence of $(11,7)$ 6-periodic perfect code.

Consider the matrix

$$H=\begin{pmatrix}1&0&0&0&0&1&1&0&0&1&1\\0&1&0&0&1&0&1&1&0&0&1\\0&0&1&0&0&1&1&1&1&0&0\\0&0&0&1&1&0&1&0&1&1&0\end{pmatrix}_{4\times 11}$$

If this matrix is considered as a parity check matrix for $(11,7)$ code, then it can be verified from the following error pattern-syndrome table that the code under discussion is $(11,7)$ 6-periodic perfect code.

Error Pattern	Syndrome
10000000000	1000
10000001000	1110
00000001000	0110
01000000000	0100
01000000100	0111
00000000100	0011
00100000000	0010
00100000010	1011
00000000010	1001
00010000000	0001

00010000001	1101
00000000001	1100
00001000000	0101
00000100000	1010
00000010000	1111

(iv) For $s=8$, the equality (3) becomes

$$2^{n-k}=2^{k_0}+2^{k_1}+2^{k_2}+2^{k_3}+2^{k_4}+2^{k_5}+2^{k_6}+2^{k_7}+2^{k_8}-8 \quad (7)$$

We now assign values to $n,k_0,k_1,k_2,k_3,k_4,k_5,k_6,k_7$ and k_8 to check the possibilities of 8-periodic perfect codes.

Case 1. For $s=8$ and for the values of the parameters $n=12$, $k_0=k_1=k_2=2$; $k_3=k_4=k_5=k_6=k_7=k_8=1$ the equality (7) gives $k=8$.

This may give rise to $(12,8)$ 8-periodic perfect code. If we consider the matrix

$$H=\begin{pmatrix}1&0&0&0&0&1&1&1&1&0&0&1\\0&1&0&0&1&0&1&1&1&1&0&0\\0&0&1&0&0&1&0&0&1&1&1&0\\0&0&0&1&1&0&0&1&1&0&1&1\end{pmatrix}_{4\times12}$$

as the parity check matrix for $(12,8)$ code then it can be verified from error pattern syndrome table that the $(12,8)$ code is 8-periodic perfect code.

Error Pattern	**Syndrome**
100000000000	1000
100000000100	1110
000000000100	0110
010000000000	0100
010000000010	0111
000000000010	0011
001000000000	0010
001000000001	1011
000000000001	1001
000100000000	0001
000010000000	0101
000001000000	1010
000000100000	1100
000000010000	1101
000000001000	1111

(v) For $s=10$, the equality (3) becomes

$$2^{n-k}=2^{k_0}+2^{k_1}+2^{k_2}+2^{k_3}+2^{k_4}+2^{k_5}+2^{k_6}+2^{k_7}+2^{k_8}+2^{k_9}+2^{k_{10}}-10 \quad (8)$$

We now examine the values to $n,k_0,k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8,k_9$ and k_{10} to check the possibilities of 10-periodic perfect codes.

Case 1. $s=10$ and for the values of the parameters $n=13$, $k_0=k_1=2, k_2=k_3=k_4=k_5=k_6=k_7=k_8=k_9=k_{10}=1$ the equality (8) gives $k=9$.

This may give rise to $(13,9)$ 10-periodic perfect code. If we consider the matrix

$$H=\begin{pmatrix}1&0&0&0&0&1&1&1&1&1&1&0&0\\0&1&0&0&1&0&0&0&1&1&1&1&0\\0&0&1&0&0&0&1&1&0&0&1&1&1\\0&0&0&1&1&1&0&1&0&1&1&0&1\end{pmatrix}_{4\times13}$$

as the parity check matrix for $(13,9)$ code then it can be verified from error pattern syndrome table that the $(13,9)$ code is 10-periodic perfect code

Error Pattern	**Syndrome**
1000000000000	1000
1000000000010	1110
0000000000010	0110
0100000000000	0100
0100000000001	0111
0000000000001	0011
0010000000000	0010
0001000000000	0001
0000100000000	0101
0000010000000	1001
0000001000000	1010
0000000100000	1011
0000000010000	1100
0000000001000	1101
0000000000100	1111

IV. Construction of PECP-Codes

In the previous section we have considered codes for all possible values of the parameters for periodicity $s=2,4,6,8$ and 10 , and we have seen that these codes are perfect codes.

For the construction of the parity check matrix of these codes, it is sufficient to construct $s-i^{th}$ and $s+i^{th}$ columns of the parity check matrix for the following reasons:

Suppose that $s-i^{th}$ and $s+i^{th}$ columns are constructed. This means that the syndromes of the first $s-i$ and last $s+i$ single error pattern and double error patterns are different. If the number of all such syndromes is deleted from the total $n-k$ tuples, we remain with exactly s, $(n-k)$ tuples. Since we are correcting only single errors in s tuples, the remaining s tuples can be taken as the columns of the parity check matrix H irrespective of their order.

For example, if we consider $s=4$ and $H=(h_{1_1}h_{1_2}h_{1_3}h_{1_4}h_{1_5}h_{1_6}h_{1_7}h_{1_8}h_{1_9}h_{1_{10}}h_{1_{11}}\ldots h_{1_n})$ then, If h_{1_1} is in error and period is 4, then $h_{1_6}^{th}$ place will be in error and so on.

In general, if $H=(h_{1_1}h_{1_2}h_{1_3}\ldots h_{1_s}h_{1_{s+2}}\ldots h_{1_{2s+2}}h_{1_{2s+3}}\ldots h_{1_{n-1}}h_{1_n})$ denotes the parity check matrix for an (n,k) s-PECP codes and if the error starts from the first place, then the other element that will be in error will appear after s-places from second place i.e. $s+2^{th}$ place.

So while constructing the parity scheck matrix H, we will first construct h_{1_1} and $h_{1_{s+2}}{}^{th}$ columns and verify that they should be linearly independent and their linear combination should also be independent of h_{1_1} and $h_{1_{s+2}}$. The remaining columns of the parity check matrix will then be arranged in between h_{1_1} and $h_{1_{s+2}}$ in any order. The matrix so constructed will be a parity check matrix of s-periodic error correcting perfect code. It is by this technique that we can write down parity check matrices of the codes other than those whose parity check matrices are given in this paper. In fact we notice that periodic error correcting perfect codes exists only if the period s is a multiple of q. We now propose some open problems:

Problem 1. In this paper, the existence of such codes has been shown only in the binary case. The existence of such codes in non-binary case is not known.

Problem 2. The existence of such codes is shown only when the period s is a multiple of q.

Problem 3. Can there be a systematic way of constructing these codes.

It is hoped that the existence of such codes may prove to be fruitful for the development of the subject as well as from application point of view.

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