Algorithms to Construct Anti Magic Labeling of Complete Graphs

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Abstract — **The study of graph labeling has focused on finding classes of graphs which admits a particular type of labeling. In this paper we proposed the algorithms to construct *antimagic labeling, (a, d)-antimagic labeling, vertex antimagic total labeling and (a, d)-vertex antimagic total labeling* of complete graphs, which is a generalization of several other types of labelings. A connected graph G(V, E) is said to be antimagic if the edges of the graph are labeled with integers 1 to |E(G)| such that every vertex has a different sum of the labels of the incident edges. A connected graph G(V, E) is said to be (a, d)-vertex antimagic total if there exists a positive integers a, d and a bijection f: V U E →{1,2,…….|V|+|E|} such that the induced mapping g_f : V→W is also a bijection, where W={w_f(x) | x ε V}={a, a+d, a+2d, ….,a+(|V|-1)d} is the set of weights of vertices in G.**

Properties of complete graphs are studied. How to construct these labelings for complete graphs are shown. Some new results are proposed from the old one.

Key words: Magic squares, Magic constant, Complete graphs, Anti Magic Labeling, (a, 1)-Vertex Anti Magic Total Labeling etc.

I. INTRODUCTION

Let G = (V, E) be a graph which is finite, simple and undirected .The graph G has a vertex set V = V (G) and edge set E = E(G) .We denote an m = |E| and n = |V |. A standard graph theoretic notation is followed. In this paper we deal only with complete graphs.

The labeling (or valuation) of a graph is any mapping that maps some set of graph elements to a set of numbers (usually positive or non negative integers). If the domain is the edge set then it is called edge labeling. If the domain is the vertex set then it is called vertex labeling. If the domain consists of both edge set and vertex set then it is called total labeling. The most complete recent survey of graph labeling is [1].

Various authors, beginning with Sedlacek [13] have introduced labelings that generalize the idea of a magic square. The magic labelings are one-to-one maps onto the appropriate set of consecutive integers starting from 1, satisfying some kind of "constant sum" property. An edge magic labeling is one in which the sum of all labels associated with a vertex is a constant independent of the choice of edge. Similarly vertex magic labelings are defined. For the first time MacDougall, Miller, Slamin and Wallis [5] introduced ***Vertex Magic Total Labeling [VMTL].*** Such VMTL is a one-to-one mapping f: V U E → {1,2,…,m+n} with the property that there is a constant k such that at any vertex x f(x)+Σf(x, y)=k, where the sum is over all vertices y adjacent to x. For any labeling the sum of the appropriate labels at a vertex is called as the weight of the vertex, denoted by $w_f(x)$; hence for VMTL we require that the weight of all vertices be the same, namely k and this number is called the magic number for the labeling. In [14] Lin and Miller proved that all complete graphs admit VMTL. They proved this by using mutually orthogonal Latin squares. The same result was proved by Krishnappa.H.K. and others [6, 7, 8] by using magic squares and the divide and conquer technique of algorithm design.

Edge Magic Total Labeling [EMTL] have been studied by Wallis, Baskoro, Miller and Slamin [11]. Krishnappa.H.K., N.K.Srinath, Ramakanth Kumar.P. and Manjunath.S proposed an algorithm to construct EMTL for $K_{m,n}$ [12].

In [2] Hartsfield and Ringel introduced the concepts of an antimagic graphs. According to them an ***antimagic labeling*** is an edge labeling of the graph with integers 1,2,….,m so that the weight at each vertex is different from the weight at each vertex. Bodendiek and Walter [3] defined the concept of an ***(a, d)-antimagic labeling*** as an edge labeling in which the vertex weights form an arithmetic progression starting from *a* and have common difference *d.*

Martin Baca, Francois Bertault and MacDougall [4] introduce the notions of the ***Vertex Antimagic Total Labeling [VATL]*** and ***(a, d)- Vertex Antimagic Total Labeling [(a, d)-VATL]***, and conjuncture that all regular graphs are (a, d)-VATL. Also they showed that all complete graphs K_n for n=2 , n>5 and except for n≡ 0 mod 4 are (a, 1)-VATL.

*In this paper we are proposing algorithms to construct the following labelings for complete graphs K_n. These algorithms uses magic squares of order n and **Super Vertex Magic Total Labeling [SVMTL]** of K_n [9, 10].*

- *Algorithm to construct Antimagic labeling of K_n.*
- *Algorithm to construct (a, 1)-Antimagic labeling of K_n.*
- *Algorithm to construct Vertex Antimagic Total labeling of K_n.*
- *Algorithm to construct (a, 2)-Vertex Antimagic Total labeling of K_n.*

Also, we showed that for K_5 it is possible to construct (a, 1)-VATL.

II. General properties, Representations and definitions

A. Basic counting

In [10] it is shown that the VMTL of K_n has the magic constant k lies between $(n(n^2 + 3))/4$ to $(n(n+1)^2/4$.

B. Duality

Given any one total labeling on a graph, it is possible to construct other total labeling from it. Let f: V U E → {1,2,………,m+n} be a one-to one map. We define the map f' on V U E by f'(x)= M+1-f(x), x ε V and f'(xy)=M+1-f(xy), xy ε E, where M=n+m. Clearly, f' is also a one-to-one map from the set V U E to {1,2,3,……,m+n}. We call f and f' are duals to each other.

C. Representation

The algorithms which we are proposing uses n x n matrix. The principle diagonal entries are used to label the vertices of the graph and the other entries are used to label the edges of the graph if the label is VATL or (a, d)-VATL. If the labeling is antimagic or (a, 1)- antimagic the diagonal entries are filled with zeros.

V_1	(V_1, V_2)	(V_1, V_3)	……..	(V_1, V_n)
(V_2, V_1)	V_2	(V_2, V_3)	……..	(V_2, V_n)
(V_3, V_1)	(V_1, V_2)	V_3	……..	(V_3, V_n)
. . . . .	. . . . .	. . . . .	. . . . .	
(V_n, V_1)	(V_n, V_2)	(V_n, V_3)	……..	V_n

Table 1: Representation table.

After constructing the representation table for K_n using our algorithm one can easily verify whether the graph is antimagic, (a, 1)-antimagic, VATL or (a, d)-VATL OR not by simply take the sum of each columns and verify.

D. Definitions

Definition 1: If there exists a one-to-one map f: E→Z^+ such that all vertices have the same weight w(v), then the graph is called ***magic*** and the map f is called a ***magic labeling*** of G.

Definition 2: If there exists a one-to-one map f: V U E→{1,2,3,…..,m+n} such that all vertices x ε V, f(x) + Σf(xy) is constant, where the sum is over all the edges which incident on x then the graph is called ***vertex magic total*** and the map f is called a ***vertex magic total labeling*** of G.

Definition 3: If there exists a one-to-one map f: V U E→{1,2,3,…..,m+n} such that all edges yx ε E, f(x) + f(xy) + f(y) is constant, then the graph is called ***edge magic total*** and the map f is called a ***edge magic total labeling*** of G.

Definition 4: If there exists a one-to-one map f: E→{1,2,3,…..,m} such that the weight at each vertex is different from the weight at any other vertex, then the graph is called ***antimagic*** and the map f is called a ***antimagic labeling*** of G.

Definition 5: A connected graph G= (V, E) is said to be ***(a, d)-antimagic*** if there exists positive integers a, d and a bijection f: E→{1,2,3,…..,m} such that W={w(v) | v ε V} = {a, a+d, a+2d,……a+(n-1)d} is the set of weights of vertices. The map f is called a ***(a, d)-antimagic labeling*** of G.

Definition 6: A bijection f: V U E→{1,2,3,…..,m+n} is called ***vertex antimagic total labeling*** of G= (V, E) if the weights of vertices wf(x), x ε V are pair wise distinct.

Definition 7: A bijection f: V U E→{1,2,3,…..,m+n} is called an ***(a, d)- vertex antimagic total labeling*** of G= (V, E) if the set of vertex weights is W={w(v) | v ε V} = {a, a+d, a+2d,…… a+(n-1)d} for some integers a and d.

*In this paper we are proposed the algorithms to construct **antimagic labeling, (a, 1)-antimagic labeling, vertex antimagic total labeling** and **(a, 2)- vertex antimagic total labeling** of complete graphs.*

III. ALGORITHMS

A. Algorithm to construct antimagic labeling of complete graphs.

Algorithm antimagic_labeling_K_n (int n)

// This algorithm takes an integer parameter n. Here n //indicates the number of vertices of complete graph // K_n. This fills the n x n matrix M by the numbers 1 to m. //Finally it computes the sum of labels of all the incident edges //at each vertex and verifies that each sums are distinct. This //algorithm uses two arrays S and temp to store the numbers 1 //to m and the computed sums at each vertex respectively.

Step-1: [Compute the number of edges]
m$\leftarrow$n(n-1)/2

Step-2: [Initialization]
Repeat for i $\leftarrow$1 to m do
S[i]$\leftarrow$i

Step-3: [Initialize the n x n matrix M to null matrix]
Repeat for i$\leftarrow$1 to n do
Repeat for j$\leftarrow$1 to n do
$M_{i,j}$$\leftarrow$0;

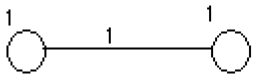
Step-4: [Fill the matrix M with the elements of the set S]
k$\leftarrow$1
Repeat for i$\leftarrow$1 to n do
Repeat for j$\leftarrow$1 to n do
if ((i j) && ($M_{i,j}$=0 && $M_{j,i}$=0)) then
$M_{i,j}$$\leftarrow$S[k]
$M_{j,i}$$\leftarrow$S[k]
k$\leftarrow$k+1

Step-5: [Compute the sum of each column and these values in temporary array *temp[]*]
val$\leftarrow$0
k$\leftarrow$1
Repeat for i$\leftarrow$1 to n do
Repeat for j$\leftarrow$1 to n do
val$\leftarrow$val+$M_{j,i}$
temp[k[$\leftarrow$val
k$\leftarrow$k+1

Step-6: [Assign the values stored in temp[] to vertices of K_n in the order v_1, v_2, v_3,……, v_n, and verify that all these values are distinct]
Repeat for i$\leftarrow$1 to n do
$v_i$$\leftarrow$temp[i]

Step-7: Stop.

Example 1: K_2 is not antimagic.



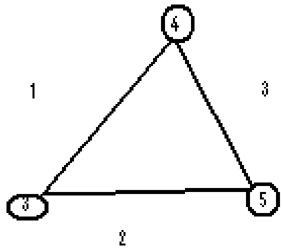
Fig(1)

K_3 is antimagic.

0	1	2
1	0	3
2	3	0
3	4	5

Table 2: Labeling of K_3

Mapping of these table entries to K_3 is shown below.



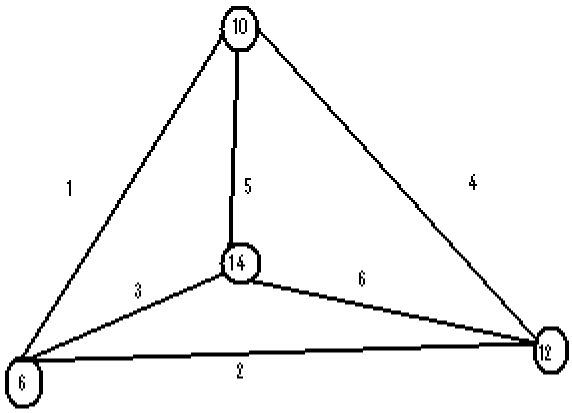
Fig(2)

K_4 is antimagic.

0	1	2	3
1	0	4	5
2	4	0	6
3	5	6	0
6	10	12	14

Table 3: Labeling of K_4

Mapping of these table entries to K_4 is shown below.



Fig(3)

B. Algorithm to construct (a, 1)-antimagic labeling of complete graphs.

This construction process is treated in 3 cases.

- (a, 1)-antimagic of K_n, where n is odd.

- (a, 1)-antimagic of K_n, where n≡ 2 mod 4.
- (a, 1)-antimagic of K_n, where n ≡0 mod 4.

a. Algorithm to construct (a, 1)-antimagic of K_n, where n is odd.

To realize (a, 1)-antimagic labeling of K_n, for n odd, we uses magic square of order n and by generalizing the magic square we obtained the desired labeling.

Algorithm (a, 1) _Antimagic_labeling_K_n (int n)
// This algorithm uses odd ordered magic square and super
// vertex magic total labeling of K_n. In [10] we
// gave an algorithm to realize SVMTL of K_n. Here we are
// realizing (a, 1)-antimagic labeling by using SVMTL of K_n.

Step-1: Construct SVMTL of Kn, for n odd.

Step-2: Find the dual of SVMTL obtained in Step-1.

Step-3: Drop the vertex labels.

Step-4: Take the sum of labels of all the incident edges at each vertex and verify that these sums forms an arithmetic progression starting with some value say, a, a+1, a+2, ……,a+(n-1).

Step-5: Stop.

Example 2: K_3 is (a, 1)-antimagic with a=3.

SVMTL of K_3

1	6	5
6	2	4
5	4	3

Table 4.1: Labeling of K_3

The dual of this labeling

6	1	2
1	5	3
2	3	4

Table 4.2: Dual Labeling of K_3

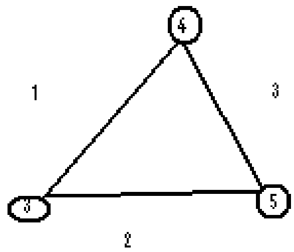
By dropping vertex labels

-	1	2
1	-	3
2	3	-

Table 4.3: (3, 1)-antimagic labeling of K_3

W={3, 4, 5} here a=3.

The mapping of these labels to edges of K_3 is as bellow.



Fig(4)

K_5 is (a, 1)-antimagic with a=20.

SVMTL and its dual of K_5 are

1	14	7	10	13
14	2	15	8	6
7	15	3	11	9
10	8	11	4	12
13	6	9	12	5

Table 5.1: Labeling of K_5

The dual of this labeling

15	2	9	6	3
2	14	1	8	10
9	1	13	5	7
6	8	5	12	4
3	10	7	4	11

Table 5.2: Dual Labeling of K_5

By dropping the vertex labels, we get

-	2	9	6	3
2	-	1	8	10
9	1	-	5	7
6	8	5	-	4
3	10	7	4	-

Table 5.3: (20, 1)-antimagic labeling of K_5

W={ 20, 21, 22, 23, 24}

The mapping of these labels to edges of K_5 is as bellow.

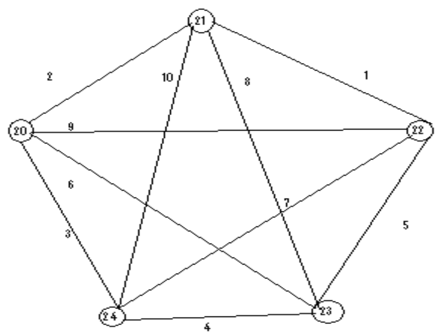


Fig (5)

b. There is no (a, 1)-antimagic labeling for Kn, where n≡2 mod 4.

In [10] we showed that there is no SVMTL for K_n, for n≡2 mod 4. Hence these complete graphs are not (a, 1)-antimagic.

c. There exists (a, 1)-antimagic labeling for Kn, where n≡0 mod 4.

In [10] we proposed the new algorithm to realize SVMTL for Kn, where n≡0 mod 4. By using this algorithm together with the algorithm given in section 3.2.1 one can obtain the (a, 1)-antimagic labeling for Kn, where n≡0 mod 4.

Example 3: K_8 is (a, 1)-antimagic with a=98.

SVMTL of K_8 is shown by the following matrix.

1	15	19	16	24	35	31	21
15	2	23	11	12	30	33	36
19	23	3	20	27	10	28	32
16	11	20	4	34	29	22	26
24	12	27	34	5	25	18	17
35	30	10	29	25	6	14	13
31	33	28	22	18	14	7	9
21	36	32	26	17	13	9	8
Magic constant=162							

Table 6.1: Labeling of K_8

The dual of the above labeling is shown by the following matrix.

36	22	18	21	13	2	6	16
22	35	14	26	25	7	4	1
18	14	34	17	10	27	9	5
21	26	17	33	3	8	15	11
13	25	10	3	32	12	19	20
2	7	27	8	12	31	23	24
6	4	9	15	19	23	30	28
16	1	5	11	20	24	28	29
Magic constant=134							

Table 6.2: Dual Labeling of K_8

After dropping the vertex labels the matrix becomes

-	22	18	21	13	2	6	16
22	-	14	26	25	7	4	1
18	14	-	17	10	27	9	5
21	26	17	-	3	8	15	11
13	25	10	3	-	12	19	20
2	7	27	8	12	-	23	24
6	4	9	15	19	23	-	28
16	1	5	11	20	24	28	-
98	99	100	101	102	103	104	105

Table 6.3: (98, 1)-antimagic labeling of K_8

C. Algorithm to construct vertex antimagic total labeling of complete graphs.

In section A above we gave an algorithm to construct antimagic labeling of K_n. This algorithm fills the n x n matrix by the numbers from 1 to n(n-1)/2, excluding the principle diagonal entries. Fill these diagonal entries using numbers [n(n-1)/2] +1, …….., [n(n-1)/2] +n starting from top-left to bottom-right.

Finally, take the sum of the entries of each column and verify that all these sums are distinct. This proves that all complete graphs admit vertex antimagic total labeling.

Example 4: K_2 is VATL

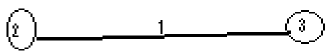


Fig (6)

K_3 is VATL

0	1	2
1	0	3
2	3	0
3	4	5

Table 7.1: Antimagic Labeling of K_3

4	1	2
1	5	3
2	3	6
7	9	11

Table 7.2: VAT Labeling of K_3

The mapping of these labels to edges of K_3 is as bellow.

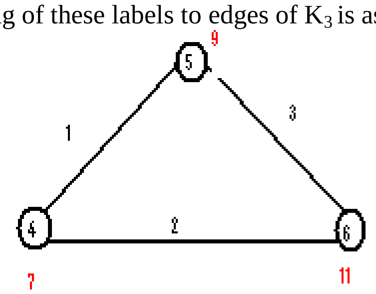


Fig (7)

K_4 is VATL

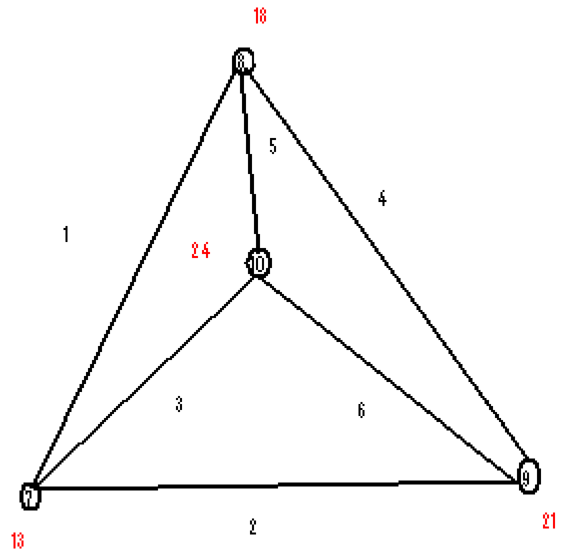
0	1	2	3
1	0	4	5
2	4	0	6
3	5	6	0
6	10	12	14

Table 8.1: Antimagic Labeling of K_4

7	1	2	3
1	8	4	5
2	4	9	6
3	5	6	10
13	18	21	24

Table 8.2: VAT Labeling of K_4

The mapping of these labels to edges of K_4 is as bellow.



Fig(8)

D. Algorithm to construct (a, 2)-vertex antimagic total labeling of complete graphs.

In [4] Martin Baca, Francois Bertault and J.A.MacDougall showed that the complete graphs K_n have (a, 1)-VATL, for n=2 and all n>5 but not n≡0 mod 4. In this paper, we are showing that K_5 is also (a, 1)-VATL and we showed that other variations of (a, d)-VATL of complete graphs.

In section 3.2 we showed that K_n is (a, 1)-antimagic for all n odd and n≡0 mod 4. Here, we are showing that these graphs are (a, 2)-VATL.

Algorithm (a, 2) _VATL_K_n (int n)
// This algorithm uses (a, 1) _Antimagic_labeling_K_n ( ) as sub
// module to compute VATL.

Step-1: Construct **(**a, 1) –antimagic labeling of K_n

Step-2: The n x n matrix obtained in step-1 consists the numbers from 1 to n(n-1)/2 excluding the diagonal entries. Use the numbers [n(n-1)/2]+1 to [n(n-1)/2]+n to fill the diagonal entries in order to get the (a, 2)-VATL.

Step-3: Take the sum of each column and verify that all these values forms an arithmetic progression a, a+2, a+4, …., a+(n-1)2.

Step-4: Stop.

Example 5: K_3 is (a, 2)-VATL

-	1	2
1	-	3
2	3	-

Table 9.1: (a, 1)-antimagic labeling of K_3

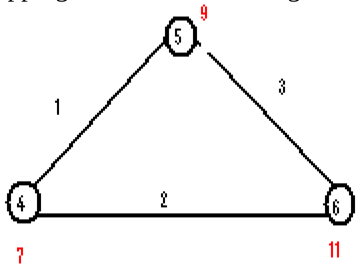
After filling the diagonals with 4, 5, 6.

4	1	2
1	5	3
2	3	6
7	9	11

Table 9.2: (a, 2)-VATL of K_3

W={7, 9, 11} is a arithmetic progression, here a=7 and d=2.

The mapping of these labels to edges of K_3 is as bellow.



Fig(9)

K_5 is (a, 2)-VATL.

-	2	9	6	3
2	-	1	8	10
9	1	-	5	7
6	8	5	-	4
3	10	7	4	-

Table 10.1: (a, 1)-antimagic labeling of K_5

11	2	9	6	3
2	12	1	8	10
9	1	13	5	7
6	8	5	14	4
3	10	7	4	15
31	33	35	37	39

Table 10.2: (a, 2)-VATL of K_5

W={31, 33, 35, 37, 39} is a arithmetic progression, here a=31 and d=2.

K_8 is (a, 2)-VATL.

-	22	18	21	13	2	6	16
22	-	14	26	25	7	4	1
18	14	-	17	10	27	9	5
21	26	17	-	3	8	15	11
13	25	10	3	-	12	19	20
2	7	27	8	12	-	23	24
6	4	9	15	19	23	-	28
16	1	5	11	20	24	28	-
98	99	100	101	102	103	104	105

Table 11.1: (a, 1)-antimagic labeling of K_8

Fill diagonals with 29, 30, 31, 32, 33, 34, 35, 36.

29	22	18	21	13	2	6	16
22	30	14	26	25	7	4	1
18	14	31	17	10	27	9	5
21	26	17	32	3	8	15	11
13	25	10	3	33	12	19	20
2	7	27	8	12	34	23	24
6	4	9	15	19	23	35	28
16	1	5	11	20	24	28	36
127	129	131	133	135	137	139	141

Table 11.2: (a, 2)-VATL of K_8

W={127, 129, 131,133, 135, 137, 139, 141} is a arithmetic progression, here a=127 and d=2.

In [4] Martin Baca, Francois Bertault and J.A.MacDougall showed that the complete graphs K_5 is not (a, 1)-VATL. In this paper we are providing one such labeling of K_5.

11	8	3	9	6
8	12	10	5	1
3	10	13	2	7
9	5	2	14	4
6	1	7	4	15
37	36	35	34	33

Table 12: (a, 1)-VATL of K_5

W={33, 34, 35, 36, 37} is a arithmetic progression, here a=33 and d=1.

Note: In the above tables the entries with green colors are vertex labels and are with red colors are the magic constants.

IV. CONCLUSION

We showed that all complete graphs K_n are

- Antimagic except for K_2.
- (a, 1)-Antimagic, for n odd and n≡0 mod 4, not for n≡2 mod 4.
- Vertex Antimagic Total Labelin.
- (a, 2)-VATL, for n odd and n≡0 mod 4, not for n≡2 mod 4.
- K_5 is (a, 1)-VATL.

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