A new type algorithm for the generalized variational inequality with multi-valued mapping

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Abstract—**In this paper, the generalized variational inequality with multi-valued mapping (GVI) is considered. To solve the problem, we first establish a global error bound estimation for GVI with the underlying multi-valued mapping being $\gamma-$strict monotone and Holder continuous. Based on this, we propose a new type of method to solve the GVI, and its global convergence is also established.**

Keywords-GVI; global error bound; algorithm; global convergence; multi-valued mapping

I. I. INTRODUCTION

We consider an algorithm for the following generalized variational inequality (abbreviated as GVI). GVI is to find vector $x^* \in C$ and $\xi^* \in F(x^*)$ such that

$$\langle \xi^*, y - x^* \rangle \geq 0, \forall y \in C, \qquad (1)$$

where C is a nonempty closed convex set in R^n, F is a multi valued mapping from C to R^n. We denote the solution set of the GVI by Ω and assume that it is nonempty throughout this paper.

In recent years, variational inequality problems have been extended in many directions via innovative techniques to study a wide class of problems arising in pure and applied sciences. The GVI plays a significant role in economics, logistics management, engineering and operation research, etc. ([1, 2]). Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature ([3-8]). Recently, Li et al. ([9]) proposes a projection algorithm for GVI with pseudo-monotone mapping and show that their algorithm converges globally, but in their paper, choosing $\xi^k \in F(x^k)$ needs solving a single-valued variational inequality and hence is computationally expensive, different to this paper, in our method, $\xi^k \in F(x^k)$ can be taken arbitrarily. Fang et al. ([10]) introduce a new projection algorithm for GVI, in which two projections need to be solved during each iteration, and establish the global convergence under suitable condition. In our method, we only need to solve a projection during each iteration. Motivated by these, we present a new type of solution method for solving the GVI.

In this paper, we first establish a global error bound estimation for GVI with the underlying multi-valued mapping being $\gamma-$strict monotone and Holder continuous. Based on this error bound, we present our method to solve the problem (1), and the global convergence is also established under milder conditions.

We end this section with some notation used in this paper. The Euclidean norm and the standard inner product of vector in the space are denoted by $\|\cdot\|$ and $\langle\cdot\rangle$, respectively. We use $dist(y,\Omega) := \inf\{\|y-x\| \,|\, x \in \Omega\}$ denote the distance from a point y to the set Ω.

II. PRELIMINARY

In this section, we review some definitions and basic results which will be used in the sequel.

Definition The Multi-valued mapping F from C into R^n is said to be:

(i) Holder continuous, if there are two constants $m > 0$, $\mu \in (0,1]$ such that

$$H(F(x), F(y)) \leq m\|x - y\|^{\mu}, \forall x, y \in C,$$

where H is Hausdorff measure.

(ii) $\gamma-$strict monotone, if there are two constants $L > 0, \gamma \in (0,1]$ such that

$$\langle \xi_1 - \xi_2, x - y \rangle \geq L\|x - y\|^{1+\gamma},$$

for any $\xi_1 \in F(x), \xi_2 \in F(y)$.

If $\mu = 1$`, F is said to be Lipchitz continuous, and if $\gamma = 1, F$ is said to be strong monotone.

Now, we give the definition of projection operator and some relate properties. Let C be a nonempty closed convex in R^n. For $x \in R^n$, we defined that

$$P_C(x) = \arg\min\{\|x - y\| \mid y \in C\},$$

where $P_C(x)$ is said to be the projection from x into C,

$P_C(\cdot)$ is said to be projection operator from R^n into C. For projection operator $P_C(\cdot)$, we have the following properties from ([2]).

Lemma 1 Let $P_C(\cdot)$ be the projection into C. Then

(i) $\langle P_C(x) - x, y - P_C(x)\rangle \ge 0, \forall x \in R^n, y \in C.$

(ii) $\|P_C(x) - P_C(y)\| \le \|x - y\|, \forall x, y \in R^n.$

For GVI, one can prove that (1) is equivalent to the fixed point problem, this result is due to Li and He ([9]). For convenience, throughout this paper, we define the projection residue vector $r(x,\xi) = \|e(x,\xi)\| = \|x - P_C(x - \beta\xi)\|$ and let $\beta > 0$ throughout this paper.

Lemma 2 $x \in C$ and $\xi \in F(x)$ solves problem (1) if and only if $r(x,\xi) = 0$.

III. THE GLOBAL ERROR BOUND FOR GVI

In order to attain the desired results on global error bound. First, we need to give the following assumption for the following analysis.

Assumption 1 For the multi-valued mapping F involved in the GVI, we assume that:

(A1) F are Holder continuous with constants $m > 0$ and $\mu \in (0,1]$.

(A2) F are $\gamma-$ strict monotone with constants $L > 0$ and $\gamma \in (0,1]$.

To establish the global error bound of GVI, we also need the following conclusion.

Lemma 2 Assumed that Assumption 1(A2) holds, if the problem (1) has solution, then it has a unique solution.

Proof Assumed that $x^* \in \Omega, \xi^* \in F(x^*)$, for any $y \in C$, we have

$$\langle \xi^*, y - x^*\rangle \ge 0. \qquad (2)$$

For any $x \in C, \xi \in F(x)$, let $y = P_C(x - \beta\xi)$ in (2), we can obtain

$$\langle \beta\xi^*, P_C(x - \beta\xi) - x^*\rangle \ge 0. \qquad (3)$$

Combining Lemma 1 (i) and $x^* \in \Omega$, we have

$$\langle P_C(x - \beta\xi) - (x - \beta\xi), x^* - P_C(x - \beta\xi)\rangle \ge 0, \quad (4)$$

Combining (3) and (4), one has

$$\begin{aligned}
&\langle P_C(x - \beta\xi) - (x - \beta\xi) - \beta\xi^*, x^* - P_C(x - \beta\xi)\rangle \\
&= \langle \beta\xi - \beta\xi^* - e(x), e(x) + x^* - x\rangle \\
&= \langle \beta(\xi - \xi^*) - e(x), e(x)\rangle + \langle \beta(\xi - \xi^*) - e(x), x^* - x\rangle \\
&= \langle \beta(\xi - \xi^*), e(x)\rangle - \langle e(x), e(x)\rangle \\
&\quad - \langle \beta(\xi - \xi^*), x - x^*\rangle + \langle e(x), x - x^*\rangle \ge 0.
\end{aligned}$$

Thus, we have

$$\langle \beta(\xi - \xi^*) + x - x^*, e(x)\rangle \ge \langle \beta(\xi - \xi^*), x - x^*\rangle. \qquad (5)$$

Moreover F is $\gamma-$strict monotone, so, there exist two constants $L > 0, \gamma \in (0,1]$ such that

$$L\beta\|x - x^*\|^{1+\gamma} \le \langle \beta(\xi - \xi^*), x - x^*\rangle. \qquad (6)$$

Combining (5) and (6), we can get

$$\begin{aligned}
&L\beta\|x - x^*\|^{1+\gamma} \\
&\le r(x,\xi)(\|\beta(\xi - \xi^*)\| + \|x - x^*\|).
\end{aligned} \qquad (7)$$

If $x \in \Omega$, combing (6) and Lemma 2, one has $x = x^*$. This completes the proof.

Based on Lemma 3, we are now ready to establish the error bound of GVI.

Theorem 1 Assumed that Assumption 1 holds, if

$$L \le (2 + \beta m)(1 + \beta m)/\beta,$$

then for any $x \in C, \xi \in F(x)$,

(i) If $r(x,\xi) \le \beta L/(\beta m + 1)$, we have

$$\left(\frac{r(x,\xi)}{2 + \beta m}\right)^{\frac{1}{\mu}} \le \|x - x^*\| \le \left(\frac{r(x,\xi)(\beta m + 1)}{L\beta}\right)^{\frac{1}{1+\gamma-\mu}}.$$

(ii) If $r(x,\xi) \ge 2 + \beta m$, we have

$$\frac{r(x,\xi)}{2 + m\beta} \le \|x - x^*\| \le \left(\frac{r(x,\xi)(m\beta + 1)}{L\beta}\right)^{\frac{1}{\gamma}}.$$

(iii) If $\beta L/(\beta m + 1) \le r(x,\xi) \le 2 + \beta m$, we have

$$\left(\frac{r(x,\xi)}{2 + m\beta}\right)^{\frac{1}{\mu}} < \|x - x^*\| < \left(\frac{r(x,\xi)(m\beta + 1)}{L\beta}\right)^{\frac{1}{\gamma}}.$$

Proof For any $x^* \in \Omega, \xi^* \in F(x^*)$, for any $x \in C$, $\xi \in F(x)$, combining (7) with Assumption 1 (A1), we have

$$\begin{aligned}
&L\beta\|x - x^*\|^{1+\gamma} \\
&\le r(x,\xi)(\beta\|\xi - \xi^*\| + \|x - x^*\|) \\
&\le r(x,\xi)(\beta m\|x - x^*\|^{\mu} + \|x - x^*\|).
\end{aligned} \qquad (8)$$

Using (8), if $\|x - x^*\| \le 1$, we can know

$$\|x - x^*\| \le \left(\frac{r(x,\xi)(m\beta + 1)}{L\beta}\right)^{\frac{1}{1+\gamma-\mu}}, \qquad (9)$$

and we can also know

$$\|x - x^*\| \ge \left(\frac{r(x,\xi)(m\beta + 1)}{L\beta}\right)^{\frac{1}{\gamma}}, \qquad (10)$$

if $\|x-x^*\| \geq 1$.

Moreover, combining Lemma 1(ii) and Lemma 2, one has

$$\begin{aligned} & r(x,\xi) \\ &= \|e(x,\xi)-e(x^*,\xi^*)\| \\ &= \|x-P_C(x-\beta\xi)-(x^*-P_C(x^*-\beta\xi^*))\| \\ &\leq \|x-x^*\|+\|P_C(x-\beta\xi)-P_C(x^*-\beta\xi^*)\| \\ &\leq \|x-x^*\|+\|x-\beta\xi-(x^*-\beta\xi^*)\| \\ &\leq 2\|x-x^*\|+\|\beta(\xi-\xi^*)\| \\ &\leq 2\|x-x^*\|+m\beta\|x-x^*\|^{\mu}. \end{aligned} \tag{11}$$

According to (11), if $\|x-x^*\| \leq 1$, we have

$$\|x-x^*\| \geq \left(\frac{r(x,\xi)}{2+m\beta}\right)^{\frac{1}{\mu}}, \tag{12}$$

and we can also obtain

$$\|x-x^*\| \geq \frac{r(x,\xi)}{2+m\beta}, \tag{13}$$

If $\|x-x^*\| \geq 1$.

Combining (9) and (12), we can get

$$\begin{aligned} &\left(\frac{r(x,\xi)}{2+m\beta}\right)^{\frac{1}{\mu}} \leq \|x-x^*\|. \\ &\leq \min\left\{\left(\frac{r(x,\xi)(m\beta+1)}{L\beta}\right)^{\frac{1}{1+\gamma-\mu}},1\right\}. \end{aligned} \tag{14}$$

Combining (10) and (13), we can also get

$$\begin{aligned} &\max\left\{\frac{r(x,\xi)}{2+m\beta},1\right\} \\ &\leq \|x-x^*\| \leq .\left(\frac{r(x,\xi)(m\beta+1)}{L\beta}\right)^{\frac{1}{\gamma}}. \end{aligned} \tag{15}$$

By (14) and (15), we have that:

If $r(x,\xi) \leq \dfrac{L\beta}{\beta m+1}$, we have $\dfrac{r(x,\xi)(m\beta+1)}{L\beta} \leq 1$, by (14), then (i) holds.

If $r(x,\xi) \geq 2+m\beta$, we can obtain $\dfrac{r(x,\xi)}{2+m\beta} \geq 1$, by (15), then (ii) holds.

If $\dfrac{L\beta}{\beta m+1} \leq r(x,\xi) \leq 2+\beta m$, combining (14) and (15), then (iii) holds. This completes the proof.

By Theorem 1, if $\mu=1, \gamma=1$, then the following result is straightforward.

Corollary 1 Assumed that F are Lipschtiz continuous with constant $m>0$ and strong monotone with constant $L>0$, then for any $x \in C, \xi \in F(x)$, we have

$$\frac{r(x,\xi)}{2+m\beta} \leq \|x-x^*\| \leq \frac{r(x,\xi)(m\beta+1)}{L\beta}.$$

IV. ALGORITHM AND CONVERGENCE

In this section, we formally state our algorithm and prove its global convergence based on the error bound results obtained in previous section. In this following, we first recall the definition of continuous multi-valued mapping:

F is said to be upper semi-continuous at $x \in C$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for any $y \in C \cap U$;

F is said to be lower semi-continuous at $x \in C$ if for any $y \in F(x)$ and any open set V containing y, there is an open set U containing x such that $F(y) \cap V \neq \emptyset$ for all $y \in C \cap U$;

F is said to be continuous at $x \in C$ if it is both upper semi continuous and lower semi-continuous at x.

If F is single-valued, then both upper semi-continuous and lower semi-continuous reduce to the continuity of F.

Now, we formally state our algorithm.

Algorithm 1

Step 0. Take $\varepsilon > 0, 0<\beta<1, L>\dfrac{m^2}{2}$, and initial point $x^0 \in C, \xi^0 \in F(x^0)$. Set $k:=0$;

Step 1. If $\|x^{k+1}-x^k\| \leq \varepsilon$, then stop, else take $\xi^k \in F(x^k)$ and go to step 2;

Step 2. Compute

$$x^{k+1}=P_C(x^k-\beta\xi^k), \tag{16}$$

increase the value of k by one, and go to step 1.

By the definition of projection operator, we can easily get that problem (16) can be equivalently reformulated as the following constrained optimization problem

$$\min (x-x^k)^{\mathrm{T}}(x-x^k)+2\beta(\xi^k)^{\mathrm{T}}(x-x^k)$$

$$\text{s.t.} \quad x \in C, \tag{17}$$

where $\beta>0$ is a constant..

We now establish the global convergence.

Theorem 2 If the multi-valued mapping F involved in the GVI is continuous with nonempty compact convex values on C and the condition of Corollary 1 holds, then the Algorithm 1 generates sequence $\{x^k\}$ globally converging to a solution of the GVI.

Proof If $x^{k+1}=x^k$, by Lemma 2, we can get that x^k is a solution of the problem (1).

In the following analysis, we assume that Algorithm 1 generates an infinite sequence. Suppose that $x^{k+1}\neq x^k$ holds and the objective function of (17) is denoted by

$$\Phi(x)=(x-x^*)^{\mathrm{T}}(x-x^*)+2\beta(\xi^*)^{\mathrm{T}}(x-x^*)$$

with $x^k=x^*,\xi^k=\xi^*$. In this following, we would prove that the sequence $\{\Phi(x^k)\}$ is monotone. To this end, we set

$$\begin{aligned}
&\Psi(k,k+1)\\
&=\Phi(x^k)-\Phi(x^{k-1})\\
&=(x^k-x^*)^{\mathrm{T}}(x^k-x^*)\\
&+2\beta(\xi^*)^{\mathrm{T}}(x^k-x^*)\\
&-(x^{k+1}-x^*)^{\mathrm{T}}(x^{k+1}-x^*)\\
&-2\beta(\xi^*)^{\mathrm{T}}(x^{k+1}-x^*)\\
&=(x^k)^{\mathrm{T}}x^k-2\langle x^*,x^k\rangle\\
&+(x^*)^{\mathrm{T}}x^*+2\beta(\xi^*)^{\mathrm{T}}(x^k-x^*)\\
&-(x^{k+1})^{\mathrm{T}}x^{k+1}+2\langle x^*,x^{k+1}\rangle\\
&-(x^*)^{\mathrm{T}}x^*-2\beta(\xi^*)^{\mathrm{T}}(x^{k+1}-x^*)\\
&=(x^k)^{\mathrm{T}}x^k-(x^{k+1})^{\mathrm{T}}x^{k+1}\\
&-2\langle x^*,x^k-x^{k+1}\rangle+2\beta\langle\xi^*,x^k-x^{k+1}\rangle\\
&=(x^k)^{\mathrm{T}}x^k-(x^{k+1})^{\mathrm{T}}x^{k+1}\\
&-2\langle x^{k+1},x^k-x^{k+1}\rangle\\
&+2\langle x^{k+1}-x^*,x^k-x^{k+1}\rangle\\
&+2\beta\langle\xi^*,x^k-x^{k+1}\rangle\\
&=(x^k-x^{k+1})^{\mathrm{T}}(x^k-x^{k+1})\\
&+2\langle x^{k+1}-x^*,x^k-x^{k+1}\rangle\\
&+2\beta\langle\xi^*,x^k-x^{k+1}\rangle.
\end{aligned}\tag{18}$$

Since (17) can be equivalently reformulated as the following variational inequality

$$\langle 2(x^{k+1}-x^k),x-x^{k+1}\rangle+2\beta\langle\xi^k,x-x^{k+1}\rangle\geq 0,\forall x\in C.\tag{19}$$

Let $x=x^*$ in (19), by (18), one has

$$\begin{aligned}
&\Psi(k,k+1)\\
&=\Phi(x^k)-\Phi(x^{k-1})\\
&\geq(x^k-x^{k+1})^{\mathrm{T}}(x^k-x^{k+1})\\
&-2\beta\langle\xi^k,x^*-x^{k+1}\rangle\\
&+2\beta\langle\xi^*,x^k-x^{k+1}\rangle\\
&=(x^k-x^{k+1})^{\mathrm{T}}(x^k-x^{k+1})\\
&+2\beta\langle\xi^k,x^k-x^*\rangle\\
&-2\beta\langle\xi^k,x^k-x^{k+1}\rangle\\
&+2\beta\langle\xi^*,x^k-x^{k+1}\rangle\\
&=(x^k-x^{k+1})^{\mathrm{T}}(x^k-x^{k+1})\\
&+2\beta\langle\xi^k,x^k-x^*\rangle\\
&-2\beta\langle\xi^k-\xi^*,x^k-x^{k+1}\rangle.
\end{aligned}\tag{20}$$

As F is strong monotone, we have

$$\begin{aligned}
&L\|x^k-x^*\|^2\\
&\leq\langle\xi^k-\xi^*,x^k-x^*\rangle\\
&=\langle\xi^k,x^k-x^*\rangle-\langle\xi^*,x^k-x^*\rangle.
\end{aligned}$$

By this fact, we can obtain

$$\begin{aligned}
&\langle\xi^k,x^k-x^*\rangle\\
&\geq L\|x^k-x^*\|^2+\langle\xi^*,x^k-x^*\rangle\\
&\geq L\|x^k-x^*\|^2.
\end{aligned}\tag{21}$$

Combining (20) and (21), we get

$$\begin{aligned}
&\Psi(k,k+1)\\
&=\Phi(x^k)-\Phi(x^{k-1})\\
&\geq(x^k-x^{k+1})^{\mathrm{T}}(x^k-x^{k+1})\\
&+2\beta L\|x^k-x^*\|^2\\
&-2\beta\langle\xi^k-\xi^*,x^k-x^{k+1}\rangle.
\end{aligned}\tag{22}$$

According to Cauchy-Schwarz inequality, we can know

$$\begin{aligned}&\langle \xi^k - \xi^*, x^k - x^{k+1} \rangle \\ &= (\xi^k - \xi^*)^{\mathrm{T}} (x^k - x^{k+1}) \\ &\le \|\xi^k - \xi^*\| \|x^k - x^{k+1}\|.\end{aligned}$$

Combining this fact and (22), we have

$$\begin{aligned}&\Psi(k, k+1) \\ &= \Phi(x^k) - \Phi(x^{k-1}) \\ &\ge \|x^k - x^{k+1}\|^2 + 2\beta L \|x^k - x^*\|^2 \\ &- 2\beta \|\xi^k - \xi^*\| \|x^k - x^{k+1}\| \\ &\ge \|x^k - x^{k+1}\|^2 + 2\beta L \|x^k - x^*\|^2 \\ &- \beta \|\xi^k - \xi^*\|^2 - \beta \|x^k - x^{k+1}\|^2 \\ &\ge \|x^k - x^{k+1}\|^2 + 2\beta L \|x^k - x^*\|^2 \\ &- \beta m^2 \|x^k - x^*\|^2 - \beta \|x^k - x^{k+1}\|^2 \\ &= (1-\beta) \|x^k - x^{k+1}\|^2 \\ &+ \beta(2L - m^2) \|x^k - x^*\|^2,\end{aligned}$$

where the last inequality is based on Lipchitz continuous. As $0 < \beta < 1, L > m^2/2$, one has $\Psi(k, k+1) > 0$, so $\{\Phi(x^k)\}$ is strictly decreasing.

According to the definition of $\Phi(x^k)$, we can know

$$\begin{aligned}&\Phi(x^k) \\ &= \|x^k - x^*\|^2 + 2\beta \langle \xi^*, x^k - x^* \rangle \qquad (23) \\ &\ge \|x^k - x^*\|^2.\end{aligned}$$

So, $\{\Phi(x^k)\}$ converges, then when $k \to \infty$, we have $\Psi(k, k+1) \to 0$ and

$$\begin{aligned}&\lim_{k\to\infty} \|x^k - x^{k+1}\| \\ &= \lim_{k\to\infty} \|x^k - P_C(x^k - \beta \xi^k)\| \\ &= \lim_{k\to\infty} r(x^k, \xi^k) = 0.\end{aligned}$$

Combining Corollary 1 with (23), one has

$$dist(x^k, \Omega) \le \frac{r(x^k, \xi^k)(\beta m + 1)}{\beta L} \to 0,$$

as $k \to \infty$. Moreover, because $\{\Phi(x^k)\}$ is convergent, so we can get $\{\Phi(x^k)\}$ is bounded, combining this fact and (23), we can also get $\{x^k\}$ is bound. So, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ and converges to $\bar{x}$. On the other hand, as F is continuous with nonempty compact convex values and $\xi^k \in F(x^k)$, then $\{\xi^k\}$ is bounded, so there exists a subsequence $\{\xi^{k_i}\}$ of $\{\xi^k\}$ and converges to $\bar{\xi}$, according to the Proposition 3.7 in [11], we have F is closed, thus $\bar{\xi} \in F(\bar{x})$.

For sequence $\{(x^{k_i}, \xi^{k_i})\}$ we can obtain

$$dist(x^{k_i}, \Omega) \le \frac{r(x^{k_i}, \xi^{k_i})(\beta m + 1)}{\beta L} \to 0$$

as $i \to \infty$. By Lemma 2, we know that $(\bar{x}, \bar{\xi})$ a solution of problem (1).

Substituting x^* in (23) by $\bar{x}$ leads to that $\Phi(x^{k_i}) \to 0$ ($i \to \infty$) as $\{\Phi(x^k)\}$ is convergent. Using (23) again, we know that $\|x^k - \bar{x}\| \to 0 (k \to \infty)$. This completes the proof.

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