Exponential stability of numerical solutions to stochastic age-structured competitive population system with diffusion

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Abstract—**This paper studies a stochastic age-dependent competitive population system with diffusion. The definition of exponential mean square stability of numerical method is introduced. It is proved that the Euler scheme is exponentially stable in mean square sense under the given conditions. An example is given for illustration.**

Keywords*：*Stochastic competitive population system; Diffusion; Euler method; Exponential stability

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1. Introduction

Stochastic differential equations have many important applications in such areas as economics, biology, finance, ecology and other sciences[1-3]. In the present investigation, the random behavior of the death and influence of external environment process are carefully incorporated into the age-dependent population equations to obtain a system of stochastic differential equations that model age-dependent population dynamics. This age-dependent population model is of theoretical interest.

The effects of the stochastic environmental noise considerations lead to stochastic age-structured population systems, which are more realistic. The study of stochastic age-structured single species system was initiated by Zhang [4]. Since then, existence, uniqueness, stability and convergence of solutions to such stochastic population system have received many attentions from several authors. For example, Li [5,6] investigated convergence of numerical solutions to stochastic age-dependent population system with Poisson jumps, diffusions and Markovian. When considering the diffusion of the population, Zhang [7] developed numerical scheme and showed the convergence of the numerical approximation solution for a stochastic age-dependent population system. In addition, Zhang [8,9] studied the existence, uniqueness and exponential stability of numerical solutions for a stochastic age-structured population system with diffusion. Wang [10] investigated convergence of the semi-implicit Euler method for stochastic age-dependent population equations with Poisson jumps.

For the problem of multi-species, Liu [11] studied discrete competitive and cooperative models of Lotka-Volterra type. Luo [12] investigated optimal birth control for predator-prey system of three species with age-structured. However, their investigations did not take stochastic factors into account. To the best of our knowledge, there haven't been any results on the topic of stochastic competitive population system of two species with diffusion.

In this paper, we shall discuss the exponential stability of stochastic partial differential equations. That is, we consider the exponential stability of stochastic age-structured competitive population systems with diffusion

$$\begin{cases} \frac{\partial P_1}{\partial t}+\frac{\partial P_1}{\partial r}-k_1(r,t)\Delta P_1+\mu_1(r,t,x)P_1 \\ \quad =-\lambda_1(r,t,x)y_2P_1+f_1(r,t,x,P_1) \\ \qquad +g_1(r,t,x,P_1)\frac{\partial W}{\partial t}, & \text{in} Q_A=(0,A)\times Q, & (1.1) \\ \frac{\partial P_2}{\partial t}+\frac{\partial P_2}{\partial r}-k_2(r,t)\Delta P_2+\mu_2(r,t,x)P_2 \\ \quad =-\lambda_2(r,t,x)y_1P_2+f_2(r,t,x,P_2) \\ \qquad +g_2(r,t,x,P_2)\frac{\partial W}{\partial t}, & \text{in } Q_A=(0,A)\times Q, & (1.2) \\ P_i(0,t,x)=\int_0^A \beta_i(r,t,x)P_i(r,t,x)dr, & \text{in}(0,T)\times\Gamma, & (1.3) \\ P_i(r,0,x)=P_{i0}(r,x), & \text{in } Q_A=(0,A)\times\Gamma, & (1.4) \\ P_i(r,t,x)=0, & \text{on } \Sigma_A=(0,A)\times(0,T)\times\partial\Gamma, & (1.5) \\ y_i(t,x)=\int_0^A P_i(r,t,x)dr, & \text{in } Q, & (1.6) \end{cases}$$

Where $(i=1,2)$, $t\in(0,T)$, $r\in(0,A)$, $x\in\Gamma$, $P_i(r,t,x)$ are the density with respect to age a of ith population at time t and in the location x, $\beta_i(r,t,x)$ and

$\mu_i(r,t,x)$ denote the fertility rate and mortality rate of females of age r at time t and in spatial position x ,respectively. Δ denotes the Laplace operator with respect to the space variable, k_i (constant) >0 are the diffusion coefficient. $\lambda_i(r,t,x)$ represent the interspecific acting functions and they are all bounded. $f_i(r,t,x,P_i)+g_i(r,t,x,P_i)\frac{dW_t}{dt}$ denote effects of external environment for population system, such as emigration and earthquake and so on.

In general, stochastic age-structures mathematical models with diffusion rarely has an explicit solution. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we will develop a numerical approximation method for stochastic age-structures population system with diffusion of the type described by Eqs. $(1.1)-(1.6)$. The numerical solution is defined by an implicit equation containing partial derivative. In particular, our results extend those of Zhang [13].

The structure of the paper is as follows: In section 2, we begin with some preliminary results, which are essential for our analysis, and introduce Euler approximation. In section 3, we give the main result that the Euler method is exponential stable in mean square sense under some conditions, and the proof of this main result is completed. In section 4, we provide an example to illustrate our result.

2. Preliminaries and Euler Approximation

Let $O=(0,A)\times\Gamma$, and

$$V\equiv\{\phi\mid\phi\in L^2(O),\frac{\partial\phi}{\partial x_i}\in L^2(O),$$

$$\text{where}\ \frac{\partial\phi}{\partial x_i}\ \text{are generalized partial derivatives}\}.$$

V is a Sobolev space. $H=L^2([0,A])$ such that

$$V\to H\equiv H\to{}'V'$$

V' is the dual space of V. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in V and V' respectively; by $<\cdot,\cdot>$ the duality product between V, V', and by $(\cdot,\cdot)$ the scalar product in H, and m is a constant such that $|x|\le m\|x\|,\forall x\in V$. For an operator $B\in L\ (M,H)$ be the space of all bounded linear operators from M into H, we denote by $\|B\|_2$ the Hilbert-Schmidt norm, i.e.

$$\|B\|_2^2=tr(BWB^T).$$

Let (Ω,F,P) be a complete probability space with a filtrations $\{F_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while F_0 contains all P -null sets).

Let $C=C([0,T];H)$ be the space of all continuous function from $[0,T]$ into H with sup-norm $\|\psi\|_C=\sup_{0\le s\le T}|\psi(s)|$, $L_V^p=L^p([0,T];V)$ and $L_H^p=L^p([0,T];H)$.

Definition 1 Let $(\Omega,F,\{F_t\},P)$ be the stochastic basis and ω_t a Wiener process. Suppose that P_{i0} $(i=1,2)$ are random variables such that $E|P_{i0}|^2<\infty$. A stochastic process P_{it} $(i=1,2)$ are said to be the solution on Ω to the stochastic age-structured population system for $t\in[0,T]$ if the following conditions are satisfied:

(1) P_{it} are F_t -measurable random variable $(i=1,2)$;

(2) $P_{it}\in I^p(0,T;V)\bigcap L^2(\Omega;C(0,T;V))$, $(i=1,2)$

$p>1$, $T>0$, where $I^p(0,T;V)$ denotes the space of all V-valued processes $(P_t)_{t\in[0,T]}$ (we will write P_t for short) measurable (from $[0,T]\times\Omega$ into V), and satisfying

$$E\int_0^T\|p_{it}\|^p\,dt<\infty\qquad(i=1,2).$$

Here $C(0,T;V)$ denotes the space of all continuous functions from $[0,T]$ to V;

(3) They satisfy the equation:

$$\begin{aligned}&<\frac{\partial P_1}{\partial t},v>+\int_0^t<\frac{\partial P_1}{\partial r},v>ds-\int_0^t<k_1(r,t)\Delta P_1,v>ds+\int_0^t<\mu_1(r,s,x)P_1,v>ds\\&=(P_{10},v)-\int_0^t<\lambda_1(r,t,x)y_2P_1,v>ds+\int_0^t<f_1(r,s,x,P_1),v>ds\\&+\int_0^t<g_1(r,s,x,P_1),v>dw(s)\end{aligned}\tag{2.1}$$

$$\begin{aligned}&<\frac{\partial P_2}{\partial t},v>+\int_0^t<\frac{\partial P_2}{\partial r},v>ds-\int_0^t<k_2(r,t)\Delta P_2,v>ds\\&+\int_0^t<\mu_2(r,s,x)P_2,v>ds\\&=(P_{20},v)-\int_0^t<\lambda_2(r,t,x)y_1P_2,v>ds\\&+\int_0^t<f_2(r,s,x,P_2),v>ds\\&+\int_0^t<g_2(r,s,x,P_2),v>dw(s)\end{aligned}\tag{2.2}$$

for all $v \in V$, $t \in [0,T]$, a.e. $w \in \Omega$, where the stochastic integrals are understood in the $It\hat{o}'s$ sense.

We consider the exponential stability of the following stochastic age-structured competitive population systems with diffusion

$$
\begin{cases}
\frac{\partial P_1}{\partial t}+\frac{\partial P_1}{\partial r}-k_1(r,t)\Delta P_1+\mu_1(r,t,x)P_1 \\
\quad =-\lambda_1(r,t,x)y_2P_1+f_1(r,t,x,P_1) \\
\qquad +g_1(r,t,x,P_1)\frac{\partial W}{\partial t}, & \text{in } Q_A=(0,A)\times Q, & (2.3)\\
\frac{\partial P_2}{\partial t}+\frac{\partial P_2}{\partial r}-k_2(r,t)\Delta P_2+\mu_2(r,t,x)P_2 \\
\quad =-\lambda_2(r,t,x)y_1P_2+f_2(r,t,x,P_2) \\
\qquad +g_2(r,t,x,P_2)\frac{\partial W}{\partial t}, & \text{in } Q_A=(0,A)\times Q, & (2.4)\\
P_i(0,t,x)=\int_0^A \beta_i(r,t,x)P_i(r,t,x)dr, & \text{in } (0,T)\times\Gamma, & (2.5)\\
P_i(r,0,x)=P_{i0}(r,x), & \text{in } Q_A=(0,A)\times\Gamma, & (2.6)\\
P_i(r,t,x)=0, & \text{on } \Sigma_A=(0,A)\times(0,T)\times\partial\Gamma, & (2.7)\\
y_i(t,x)=\int_0^A P_i(r,t,x)dr, & \text{in } Q, & (2.8)
\end{cases}
$$

A is the maximal age of the population species, so

$$P_i(r,t,x)=0, \forall r \ge A.$$

Let $\Delta t=\frac{T}{N}$, for system (2.3)-(2.8) the discrete approximate solutions on $t=0,\Delta t$, $2\Delta t,\cdots,N\Delta t$ are defined by the iterative scheme

$$
\begin{aligned}
&Q_{1t}^{n+1}-Q_{1t}^{n}-\frac{\partial Q_{1t}^{n}}{\partial r}\Delta t-k_1(r,t)\Delta Q_{1t}^{n}\Delta t+\mu_1(r,t,x)Q_{1t}^{n}\Delta t \\
&=-\lambda_1(r,t,x)y_2Q_{1t}^{n}\Delta t+f_1(r,t,x,Q_{1t}^{n})\Delta t+g_1(r,t,x,Q_{1t}^{n})\Delta W_n,
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&Q_{2t}^{n+1}-Q_{2t}^{n}-\frac{\partial Q_{2t}^{n}}{\partial r}\Delta t-k_2(r,t)\Delta Q_{2t}^{n}\Delta t+\mu_2(r,t,x)Q_{2t}^{n}\Delta t \\
&=-\lambda_2(r,t,x)y_1Q_{2t}^{n}\Delta t+f_2(r,t,x,Q_{2t}^{n})\Delta t+g_2(r,t,x,Q_{2t}^{n})\Delta W_n,
\end{aligned}
\tag{2.10}
$$

here, Q_{it}^n are the approximation to $P_i(r,t_n,x)$, for $t_n=n\Delta t$, the time increment is $\Delta t=\frac{T}{N}<<1$, and the Brownian motion increment is $\Delta W_n=W(t_{n+1})-W(t_n)$.

For convenience, we shall extend the discrete numerical solution to continuous time. We first define the step function

$$Z_{it}\equiv Z_i(r,t,x)=\sum_{k=0}^{N-1}Q_{it}^k 1_{[k\Delta t,(k+1)\Delta t)},(i=1,2) \tag{2.11}$$

where 1_G is the indicator function for the set G. Then we define

$$
\begin{aligned}
&Q_{1t}-P_{10}+\int_0^t\frac{\partial Q_{1s}}{\partial r}ds+\int_0^t\mu_1(r,s,x)Z_{1s}ds+\int_0^t\lambda_1(r,t,x)y_2Z_{1s}ds \\
&=\int_0^t k_1(r,s)\Delta Q_{1s}ds+\int_0^t f_1(r,s,x,Z_{1s})ds+\int_0^t g_1(r,s,x,Z_{1s})dW_s,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&Q_{2t}-P_{20}+\int_0^t\frac{\partial Q_{2s}}{\partial r}ds+\int_0^t\mu_2(r,s,x)Z_{2s}ds+\int_0^t\lambda_2(r,t,x)y_1Z_{2s}ds \\
&=\int_0^t k_2(r,s)\Delta Q_{2s}ds+\int_0^t f_2(r,s,x,Z_{2s})ds+\int_0^t g_2(r,s,x,Z_{2s})dW_s,
\end{aligned}
\tag{2.13}
$$

with $Q_{i0}=P_i(r,0,x)$, $Q_{it}=Q_i(r,t,x)$. They are straightforward to check that

$$Z_i(r,t_k,x)=Q_{it}^k=Q_i(r,t_k,x),(i=1,2).$$

First, we state the assumptions about the stochastic age-dependent population system with diffusion that will be considered:

(i) $f_i(r,t,x,0)=0$, $g_i(r,t,x,0)=0$; $(i=1,2)$.

(ii)(Lipschitz condition) there exists a positive constant K such that $p_1,p_2\in C$

$$
\begin{aligned}
&|f_i(r,t,x,p_1)-f_i(r,t,x,p_2)|\vee\|g_i(r,t,x,p_1)-g_i(r,t,x,p_2)\|_2 \\
&\le K|p_1-p_2|, a.e.t;
\end{aligned}
\tag{2.14}
$$

(iii) $\mu_i(r,t,x)$, $\beta_i(r,t,x)$, $\lambda_i(r,t,x)$ and $k_i(r,t)$ $(i=1,2)$ are continuous in $\bar{Q}$ such that

$$0\le\mu_i(r,t,x)\le\bar{\mu}<\infty,\quad 0\le\beta_i(r,t,x)\le\bar{\beta}<\infty,$$

$$0\le\lambda_i(r,t,x)\le\bar{\lambda}<\infty,\quad k_0\le k_i(r,t)\le\bar{k}. \tag{2.15}$$

(iv) $y_i(t,x)$ $(i=1,2)$ are bound in Q such that $y_i\le\bar{y}$.

Definition 2 Suppose that P_{i0} are random variable such that $E|P_{i0}|^2<\infty$. For a given step size $\Delta>0$, a numerical method is said to be exponentially stable in mean square on

Eqs.(2.3)-(2.8) if there is a pair of positive constants γ and $\bar{N}$, such that with initial data P_{i0},

$$E\,|\,Q_{it}^{n}\,|^2 \le \bar{N}E\,|\,P_{i0}\,|^2\; e^{-\gamma n\Delta}, \quad \forall n = 0,1,2,\cdots, \quad (i=1,2). \tag{2.16}$$

3. THE MAIN RESULTS

In this section, we shall provide some lemmas which are necessary for the proof of our result. Because Q_{it} are the discrete numerical solution of Eqs. (2.3)-(2.8), we first study properties of Q_{it}.

Lemma 3.1. Under assumptions (i)–(iv), for any $T>0$,

$$\sup_{0\le t\le T} E\,|\,Q_{1t}\,|^2 \le C_{1T}, \quad \sup_{0\le t\le T} E\,|\,Q_{2t}\,|^2 \le C_{2T} \tag{3.1}$$

where $C_{iT},(i=1,2)$ are positive constant independent of Δ, but they depend on $Q_{i0},(i=1,2)$ and T

Proof. From Eqs. (2.1) and (2.12), applying Itô formula to $|\,Q_{1t}\,|^2$ yields.

$$\begin{aligned}
&|\,Q_{1t}\,|^2 \\
&=|\,Q_{10}\,|^2 + 2\int_0^t\int_O [(-\frac{\partial Q_{1s}}{\partial r} + k_1(r,t)\Delta Q_{1s})Q_{1s}]dxdrds \\
&-2\int_0^t\int_O \mu_1(r,s,x)Z_{1s}Q_{1s}dxdrds \\
&-2\int_0^t\int_O \lambda_1(r,t,x)y_2 Z_{1s}Q_{1s}dxdrds \\
&+2\int_0^t\int_O f_1(r,s,x,Z_{1s})Q_{1s}dxdrds \\
&+2\int_0^t\int_O g_1(r,s,x,Z_{1s})Q_{1s}dxdrdW_s \\
&+\int_0^t \|g_1(r,s,x,Z_{1s})\|_2^2 ds.
\end{aligned}$$

Since

$$\begin{aligned}
&-\int_o^t\int_O \frac{\partial Q_{1s}}{\partial r}Q_{1s}dxdrds \\
&=-\frac{1}{2}\int_0^t\int_\Gamma [Q_1^2(A,s,x)-Q_1^2(0,s,x)]dxds \\
&=\frac{1}{2}\int_0^t\int_\Gamma (\int_0^A \beta_1^2(r,s,x)Q_1(r,s,x)dr)^2 dxds,
\end{aligned}$$

by (iii) and Holder inequality, we have

$$-2\int_0^t\int_O \frac{\partial Q_{1s}}{\partial r}Q_{1s}dxdrds \le A\bar{\beta}^2\int_0^t |\,Q_{1s}\,|^2\, ds.$$

However, by (iii), we have

$$\begin{aligned}
&\int_0^t\int_O k_1(r,s)\Delta Q_{1s}Q_{1s}dxdrds \\
&=-\int_0^t\int_O k_1(r,s)\nabla Q_{1s}\cdot\nabla Q_{1s}dxdrds. \\
&\le -k_0\int_0^t \|\,Q_{1s}\,\|^2\, ds.
\end{aligned}$$

Therefore, we get that

$$\begin{aligned}
&|\,Q_{1t}\,|^2 \le |\,Q_{10}\,|^2 + A\bar{\beta}^2\int_0^t |\,Q_{1s}\,|^2\, ds - 2k_0\int_0^t \|\,Q_{1s}\,\|\,ds \\
&+2\bar{\mu}\int_0^t |\,Q_{1s}\,\|\,Z_{1s}\,|\,ds + 2\bar{\lambda}\bar{y}\int_0^t |\,Q_{1s}\,\|\,Z_{1s}\,|\,ds \\
&+\int_0^t |\,f_1(r,s,x,Z_{1s})\,|^2\, ds + \int_0^t \|\,g_1(r,s,x,Z_{1s})\,\|_2^2\, ds \\
&+\int_0^t |\,Q_s\,|^2\, ds + 2\int_0^t\int_O g_1(r,s,x,Z_{1s})Q_{1s}dxdW_s dr.
\end{aligned}$$

Now, it follows that for any $t\in[0,T]$

$$\begin{aligned}
&E\sup_{0\le s\le t}|\,Q_{1s}\,|^2 \\
&\le E|Q_{10}|^2 + A\bar{\beta}^2\int_0^t E\sup_{0\le s\le t}|Q_{1s}|^2 ds \\
&-2k_0\int_0^t E\sup_{0\le s\le t}|Q_{1s}|^2 ds \\
&+\bar{\mu}\int_0^t E\sup_{0\le s\le t}|\,Q_{1s}\,|^2\, ds \\
&+\bar{\mu}\int_0^t E\sup_{0\le s\le t}|\,Z_{1s}\,|^2\, ds \\
&+\bar{\lambda}\bar{y}\int_0^t E\sup_{0\le s\le t}|\,Q_{1s}\,|^2\, ds \\
&+\bar{\lambda}\bar{y}\int_0^t E\sup_{0\le s\le t}|\,Z_{1s}\,|^2\, ds \\
&+\int_0^t E\,|\,f_1(r,s,x,Z_{1s})\,|^2\, ds \\
&+\int_0^t E\,\|\,g_1(r,s,x,Z_{1s})\,\|_2^2\, ds \\
&+2E\sup_{0\le s\le t}\int_0^s\int_O Q_{1r}g_1(r,\tau,x,Z_{1\tau})drdxdW_\tau.
\end{aligned}$$

Using condition (ii) yields

$$
\begin{aligned}
&E\sup_{0\le s\le t}|Q_{1s}|^2\\
&\le E|Q_{10}|^2+(A\overline{\beta}^2-2k_0+\overline{\mu}+\overline{\lambda}\overline{y})\int_0^t E\sup_{0\le s\le t}|Q_{1s}|^2\,ds \quad (3.2)\\
&+(\overline{\mu}+2K^2+\overline{\lambda}\overline{y})\int_0^t|Z_{1s}|^2\\
&+2E\sup_{0\le s\le t}\int_0^s\int_O Q_{1r}g_1(r,\tau,x,Z_{1\tau})drdxdW_\tau.
\end{aligned}
$$

By Burkholder-Davis-Gundy's inequality (see, for example, [14]), we have

$$
\begin{aligned}
&E[\sup_{0\le s\le t}\int_0^s\int_O Q_{1r}g_1(r,\tau,x,Z_{1\tau})dW_\tau]\\
&\le 3E[\sup_{0\le s\le t}|Q_{1s}|(\int_0^t\|g_1(r,s,x,Z_{1s})\|_2^2\,ds)^{1/2}]\\
&\le\frac{1}{4}E[\sup_{0\le s\le t}|Q_{1s}|^2]+K_1\int_0^t\|g_1(r,s,x,Z_{1s})\|_2^2\,ds\\
&\le\frac{1}{4}E[\sup_{0\le s\le t}|Q_{1s}|^2]+K_1\cdot K^2\int_0^t E|Z_{1s}|^2\,ds,
\end{aligned} \quad (3.3)
$$

for some positive constant $K_1>0$. Thus, it follows from (3.2) and (3.3)

$$
\begin{aligned}
E\sup_{0\le s\le t}|Q_{1s}|^2\le 2E|Q_{10}|^2&+2(A\overline{\beta}^2-2k_0+2\overline{\mu}+2K^2+2K_1K^2\\
&+2\overline{\lambda}\overline{y})\int_0^t E\sup_{0\le r\le s}|Q_{1r}|^2ds\quad \forall t\in[0,T].
\end{aligned}
$$

In the same proof way, we can

$$
\begin{aligned}
E\sup_{0\le s\le t}|Q_{2s}|^2\le 2E|Q_{20}|^2&+2(A\overline{\beta}^2-2m^2k_0+2\overline{\mu}+2K^2+2K_1K^2\\
&+2\overline{\lambda}\overline{y})\int_0^t E\sup_{0\le r\le s}|Q_{2r}|^2ds\quad \forall t\in[0,T].
\end{aligned}
$$

Now, Gronwall's lemma obviously implies the required result. The proof is complete.

Lemma 3.2. Under the assumptions (i)-(iv), for any $T>0$,

$$E\sup_{0\le t\le T}|Q_{1t}-Z_{1t}|^2\le C_{21}\Delta t\sup_{t\in[0,T]}E|Q_{1s}|^2$$

$$E\sup_{0\le t\le T}|Q_{2t}-Z_{2t}|^2\le C_{22}\Delta t\sup_{t\in[0,T]}E|Q_{2s}|^2. \quad (3.4)$$

Proof. For $\forall t\in[0,T]$, there exists an integer k such that $t\in[k\Delta t,(k+1)\Delta t)$. We have

$$
\begin{aligned}
&Q_{1t}-Z_{1t}=Q_{1t}-Q_{1t}^k\\
&=-\int_{k\Delta t}^t\frac{\partial Q_{1s}}{\partial r}ds+\int_{k\Delta t}^t k_1(r,s)\Delta Q_{1s}ds\\
&\quad-\int_{k\Delta t}^t\mu_1(r,s,x)Z_sds-\int_{k\Delta t}^t\lambda_1(r,t,x)y_2Z_{1s}ds\\
&\quad+\int_{k\Delta t}^t f_1(r,s,x,Z_{1s})ds+\int_{k\Delta t}^t g_1(r,s,x,Z_{1s})dW_s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|Q_{1t}-Z_{1t}|^2\\
&\le 6|\int_{k\Delta t}^t\frac{\partial Q_{1s}}{\partial r}ds|^2+6|\int_{k\Delta t}^t k_1(r,s)\Delta Q_{1s}ds|^2\\
&+6|\int_{k\Delta t}^t\mu_1(r,s,x)Z_{1s}ds|^2+6|\int_{k\Delta t}^t\lambda_1(r,t,x)y_2Z_{1s}ds|^2\\
&+6|\int_{k\Delta t}^t f_1(r,s,x,Z_{1s})ds|^2+6|\int_{k\Delta t}^t g_1(r,s,x,Z_{1s})dW_s|^2.
\end{aligned}
$$

Now, the Cauchy-Schwarz inequality and the assumptions (i)-(iv) give

$$
\begin{aligned}
&|Q_{1t}-Z_{1t}|^2\\
&\le 6\Delta t\int_{k\Delta t}^t|\frac{\partial Q_{1s}}{\partial r}|^2ds+6\overline{k}^2\Delta t\int_{k\Delta t}^t|\Delta Q_{1s}|^2ds\\
&+6\overline{\mu}^2\Delta t\int_{k\Delta t}^t|Z_{1s}|^2\,ds+6\overline{\lambda}^2\overline{y}_1^2\Delta t\int_{k\Delta t}^t|Z_{1s}|^2\,ds\\
&+6\Delta t\int_{k\Delta t}^t|f_1(r,s,x,Z_{1s})|^2\,ds+6|\int_{k\Delta t}^t g_1(r,s,x,Z_{1s})dW_s|^2\\
&\le 6\Delta t\int_{k\Delta t}^t|\frac{\partial Q_{1s}}{\partial r}|^2ds+6\overline{k}^2\Delta t\int_{k\Delta t}^t|\Delta Q_{1s}|^2ds+6\overline{\mu}^2\Delta t\int_{k\Delta t}^t|Z_{1s}|^2ds\\
&+6\overline{\lambda}^2\overline{y}_1^2\Delta t\int_{k\Delta t}^t|Z_{1s}|^2\,ds+5K^2\Delta t\int_{k\Delta t}^t|Z_{1s}|^2\,ds+5|\int_{k\Delta t}^t g_1(r,s,x,Z_{1s})dW_s|^2,
\end{aligned}
$$

whence applying the Burkholder-Davis-Gundy inequality and condition (ii) leads to

$$E\sup_{t\in[0,T]}|\int_{k\Delta t}^t g_1(r,s,x,Z_{1s})dW_s|^2\le C_3\int_{k\Delta t}^t E\sup_{t\in[0,T]}|Z_{1s}|^2\,ds,$$

where C_3 is a constant. Because the differential operator $\frac{\partial}{\partial r}$ and Laplace operator Δ are bounded linear operator , we obtain

$$E\sup_{t\in[0,T]}|Q_{1t}-Z_{1t}|^2 \le 6C_4\Delta t\sup_{t\in[0,T]}E|Q_{1s}|^2 + 6(\bar{\mu}^2\Delta t+\bar{\lambda}^2\bar{y}_1^2\Delta t+K^2\Delta t+C_3)\Delta t\sup_{t\in[0,T]}E|Q_{1s}|^2,$$

where C_4 is a constant.

As the same as above, we can also get

$$E\sup_{t\in[0,T]}|Q_{2t}-Z_{2t}|^2 \le 6C_4\Delta t\sup_{t\in[0,T]}E|Q_{2s}|^2 + 6(\bar{\mu}^2\Delta t+\bar{\lambda}^2\bar{y}_1^2\Delta t+K^2\Delta t+C_3')\Delta t\sup_{t\in[0,T]}E|Q_{2s}|^2,$$

The result (3.4) is obtained.

We are now in a position to prove a strong convergence result

Lemma 3.3. Under assumptions (i)-(iv), for any $T>0$

$$\sup_{0\le t\le T}E|Q_{1t}-P_{1t}|^2\le C_{1T}\Delta t\sup_{t\in[0,T]}E|Q_{1s}|^2,$$

$$\sup_{0\le t\le T}E|Q_{2t}-P_{2t}|^2\le C_{2T}\Delta t\sup_{t\in[0,T]}E|Q_{2s}|^2. \qquad (3.5)$$

where $C_{iT}(i=1,2)$ are independent of Δt, but it depends on T.,

Proof. Combining (2.3) with (2.12) has

$$\begin{aligned}P_{1t}-Q_{1t} &= -\int_0^t\frac{\partial(P_{1s}-Q_{1s})}{\partial r}ds+\int_0^t k_1(r,t)(\Delta P_{1s}-\Delta Q_{1s})ds\\ &-\int_0^t\mu_1(r,s,x)(P_{1s}-Z_{1s})ds-\int_0^t\lambda_1(r,t,x)y_2(P_{1s}-Z_{1s})ds\\ &+\int_0^t(f_1(r,s,x,P_{1s})-f_1(r,s,x,Z_{1s}))ds\\ &+\int_0^t(g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s}))dW_s.\end{aligned}$$

Therefore using Itô formula, along with the Cauchy-Schwarz inequality and (ii) yields,

$$\begin{aligned}&|P_{1t}-Q_{1t}|^2\\ &=-2\int_0^t<P_{1s}-Q_{1s},\frac{\partial(P_{1s}-Q_{1s})}{\partial r}>ds\\ &+2\int_0^t<k_1(r,s)(P_{1s}-Q_{1s}),\Delta P_{1s}-\Delta Q_{1s}>ds\\ &-2\int_0^t<P_{1s}-Q_{1s},\mu_1(r,s,x)(P_{1s}-Z_{1s})>ds\\ &-2\int_0^t<P_{1s}-Q_{1s},\lambda_1(r,s,x)y_2(P_{1s}-Z_{1s})>ds\\ &+2\int_0^t<P_{1s}-Q_{1s},f_1(r,s,x,P_{1s})-f_1(r,s,x,Z_{1s})>ds\\ &+\int_0^t\|g_1(r,s,x,P_{1s})-g(r,s,x,Z_{1s})\|_2^2ds\\ &+2\int_0^t(P_{1s}-Q_{1s},(g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s}))dW_t\\ &\le A\bar{\beta}^2\int_0^2|P_{1s}-Q_{1s}|^2ds-2k_0\int_0^2\|P_{1s}-Q_{1s}\|^2ds\\ &+2\bar{\mu}\int_0^t|P_{1s}-Z_{1s}\|P_{1s}-Q_{1s}|ds+2\bar{\lambda}\bar{y}\int_0^t|P_{1s}-Z_{1s}\|P_{1s}-Q_{1s}|ds\\ &+2K\int_0^t|P_{1s}-Q_{1s}\|P_{1s}-Z_{1s}|ds+K^2\int_0^t|P_{1s}-Z_{1s}|^2ds\\ &+2\int_o^t(P_{1s}-Q_{1s},g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s}))dW_s\end{aligned}$$

Hence, for any $t\in[0,T]$

$$\begin{aligned}&E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2\\ &\le(A\beta^2+\bar{\mu}+\bar{\lambda}\bar{y}+K-2k_0)\int_0^T E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2dt\\ &+(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2)E\int_0^T|P_{1t}-Z_{1t}|^2dt\\ &+2E\sup_{t\in[0,T]}\int_0^t(P_{1s}-Q_{1s},(g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s}))dW_s).\end{aligned} \qquad (3.6),$$

By Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}&E\sup_{t\in[0,T]}\int_0^t(P_{1s}-Q_{1s},(g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s}))dW_s)\\ &\le k_1E[\sup_{0\le t\le T}|P_{1t}-Q_{1t}|\times\\ &(\int_0^T\|g_1(r,s,x,P_{1s})-g_1(r,s,x,Z_{1s})\|_2^2ds)^{1/2}]\\ &\le\frac{1}{4}E[\sup_{0\le t\le T}|P_{1t}-Q_{1t}|^2+K_1\int_0^T E|P_{1t}-Z_{1t}|^2dt,\end{aligned} \qquad (3.7)$$

where k_1, and K_1 are two positive constants. Therefore inserting (3.7) into (3.6) has

$$E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2$$
$$\le(A\bar{\beta}^2+\bar{\mu}+\bar{\lambda}\bar{y}+K-2k_0)\int_0^T E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2dt$$
$$+(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2+2K_1)\int_0^T|P_{1t}-Z_{1t}|^2\,dt$$
$$+\frac{1}{2}E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2.$$

Hence,

$$E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2$$
$$\le 2(A\bar{\beta}^2+\bar{\mu}+\bar{\lambda}\bar{y}+K-2k_0)\int_0^T E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2dt$$
$$+2(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2+2K_1)\int_0^T|P_{1t}-Z_{1t}|^2\,dt$$
$$\le 2(A\bar{\beta}^2+\bar{\mu}+\bar{\lambda}\bar{y}+K-2k_0)\int_0^T E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2\,dt$$
$$+4(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2+2K_1)\int_0^T(|Q_{1t}-Z_{1t}|^2+|P_{1t}-Q_{1t}|^2)dt$$
$$\le 2(A\bar{\beta}^2+3\bar{\mu}+3\bar{\lambda}\bar{y}+3K+2K^2+4K_1-2k_0)\int_0^T E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2\,ds$$
$$+4(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2+2K_1)\int_0^T|Q_{1t}-Z_{1t}|^2\,dt.$$

Applying Lemma 3.2 we obtain a bound of the form

$$E\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2 \quad\le\quad D_{11}\Delta t+D_{12}\int_0^T E\sup_{s\in[0,t]}|P_{1s}-Q_{1s}|^2\,ds,$$

where

$$D_{11}=4(\bar{\mu}+\bar{\lambda}\bar{y}+K+K^2+2K_1)TC_{21}\sup_{t\in[0,T]}E|Q_{1s}|^2$$

$$D_{12}=2(A\bar{\beta}^2+3\bar{\mu}+3\bar{\lambda}\bar{y}+3K+2K^2+4K_1-2k_0).$$

By applying the Gronwall inequality, we have the following inequality

$$E\left(\sup_{t\in[0,T]}|P_{1t}-Q_{1t}|^2\right)\le D_{11}\Delta t\exp(D_{12}T).$$

In the same proof procedure, we can get

$$E\left(\sup_{t\in[0,T]}|P_{2t}-Q_{2t}|^2\right)\le D_{21}\Delta t\exp(D_{22}T)$$

where

$$D_{21}=D_{11},$$

$$D_{22}=D_{12}.$$

By Lemma 3.1, (3.5) is obtained. The proof is proved.

Lemma 3.4. Under assumptions (i)–(iv), the trivial solutions of Eqs.(2.3)-(2.8) are exponentially stable in mean square. That is, there is a pair of positive constants λ and M such that, for any P_0

$$E|P_1(t)|^2\le ME|P_0|^2e^{-\lambda t},\quad \forall t\ge 0,$$

$$E|P_2(t)|^2\le ME|P_0|^2e^{-\lambda t},\quad \forall t\ge 0. \tag{3.8}$$

The proof of this lemma is an analogous to that of Theorem in [15].

Now we are in a position to prove the main result: Theorem 3.5.

Theorem 3.5. Under assumptions (i)–(iv), the Euler method applied to Eqs.(2.3)-(2.8) is exponentially stable in mean square.

The proof of this Theorem is an analogous to that of Theorem 2.2 in [14].

4. An Example

Consider the following stochastic age-structured population system with diffusion

$$\begin{cases}
\frac{\partial P_1}{\partial t}+\frac{\partial P_1}{\partial r}-r^2\Delta P_1+2x\frac{1}{(1-r)^2}P_1 \\
\quad =-(x)^2y_2P_1-txP_1+P_1dB_t, & in\,(0,1)\times(0,T)\times(0,1), \quad (4.1)\\
\frac{\partial P_2}{\partial t}+\frac{\partial P_2}{\partial r}-2r^2\Delta P_2+2x\frac{1}{(1-r)^2}P_2 \\
\quad =-(2x)^2y_1P_2-2txP_2+P_2dB_t, & in\,(0,1)\times(0,T)\times(0,1), \quad (4.2)\\
P_i(0,t,x)=\int_0^1\frac{x}{(1-r)^2}P_i(r,t,x)dr, & \text{in } (0,T)\times(0,1), \quad (4.3)\\
P_i(r,0,x)=x\exp(-\frac{1}{1-r}), & \text{in } (0,1)\times(0.1), \quad (4.4)\\
P_i(r,t,x)=0, & \text{on } \Sigma=(0,1)\times(0,T)\times\{0,1\}, \quad (4.5)\\
y_i(t,x)=\int_0^1P_i(r,t,x)dr, & \text{in} Q, \quad (4.6)
\end{cases}$$

here B_t is a real standard Brownian motion (so, $M = R$ and $W = 1$). We can set this problem in our formulation by taking $H = L^2([0,1]\times[0,1])$, $V = W_0^1([0,1]\times[0,1])$ (a Sobolev space with elements satisfying the boundary conditions above),

$$M = R,\ k_i(r,t,x) = ir^2,\ \mu_i(r,t,x) = \frac{2x}{(1-r)^2},$$
$$\beta_i(r,t,x) = \frac{x}{(1-r)^2},\ f_i(r,t,x,P_i) = -itxP_i$$

and

$$g_i(r,t,P_i) = P_i,\ P_i(r,0,x) = x\exp(-\tfrac{1}{1-r}),\ (i = 1,2).$$

Clearly, the operators f_i and g_i satisfy conditions (i) and (ii), $k_i(t,x)$, $\mu_i(r,t,x)$, $\beta_i(r,t,x)$ and $\lambda_i(r,t,x)$ satisfy condition (iii), $y_i(t,x)$ satisfy condition (iv). Consequently, the numerical solutions of (4.1)-(4.6) are exponential stability for any $(r,t,x)\in(0,1)\times(0,T)\times(0,1)$ in the sense of Theorem 3.5.

Take $T = 1$, $x = \frac{1}{2}$ in Eqs.(4.1)-(4.6). We give the pictures below with fixed step sizes $\Delta t = 0.005$, $\Delta r = 0.05$. Six pictures are numerical simulations of the stochastic age-structured competitive system equations with 100, 1000 and 10000 experiments respectively. (Figure1.)

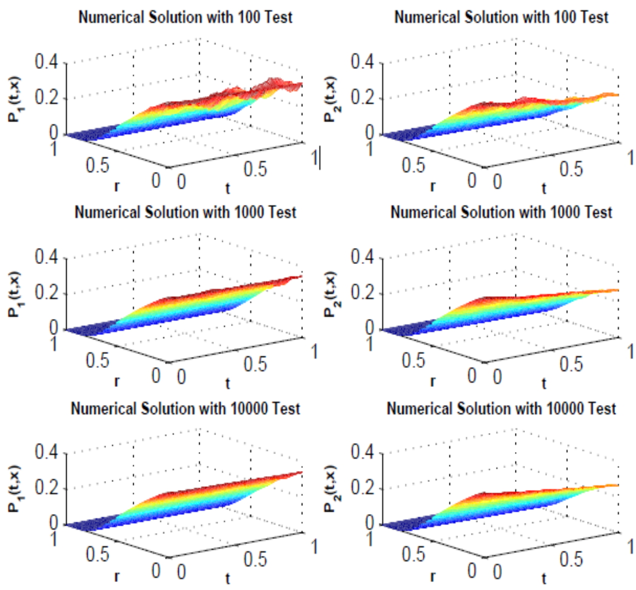

Figure 1. Exponential stability of numerical solutions to stochastic age-structured competitive population system with diffusion

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