Identification of linear system in random time

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Abstract— **In order to intelligent control in future the system should be tested. The classical tasks of identification are modeled (realized) for established number (interval, horizon) of test. In this paper the above problem is investigated for a random interval. To identification was used a conditional entropy, which represents a measure of system uncertainty. Additionally in this case the horizon is modeled by a random variable with a finite number of events. The general aim of control is the system identification at minimum cost. This problem is reduced to the task of optimal control with established finite horizon.**

Keywords - linear system, conditional entropy, optimal control, identification

I. INTRODUCTION

The problems of adaptive control, active learning, system identification and understanding, image recognizing are widely described in the literature (the first publications appeared half a century ago [2], [7], [10], [11]). To control, manage, steer, operate effectively, one should get to know the dynamic properties of system or object. Sometimes the system parameters are not known, thus making the tests they should be identified as accurately as possible. The most problems of input-output systems identification are based on a finite number of experiments, tests (see e.g. [1], [3], [5], [9], [13], [15], [16], [17]). For the determined number of tests these problems are reduced to the classical task of adaptive control. The optimal control process in this case has a dual nature - optimization of performance criterion and increasing information about the unknown parameter of the system.

Sometimes the environment does not allow us to make a established number of tests or the time of system working (functioning) is not fixed exactly due to the changing conditions (for example in different weather conditions the ship, object travels this same distance (route) at various time intervals). The question arises how to control the system for a random horizon states independent? How to make the task and the set of control laws if the horizon is not known?

In the considered case the system dynamic is known but the system parameters and the control horizon are unknown. To identify the system the conditional entropy was used. The aim of control is to minimize the costs of control (learning) and losses associated with ignorance of system, which depend on the conditional entropy of the system parameters. The horizon of control is modeled by a random variable state independent, but the parameters are modeled by a random vector. The solving of this problem is based on the construction of substitute task with the finite horizon, where the functionals of losses and heredities must be modified (see e.g. [6], [12]). In both cases (primary and auxiliary problems) the aim of control is the same but has different forms.

The organization of paper is as follows. The section 2 presents the problem of optimal control with random horizon and the reducing it to the task with established horizon. The conditional entropy is given in sections 3. The optimal control for systems identification with costs is provided in next sections suitably. The simple controls application of the proposed approach is illustrated on two examples.

II. OPTIMAL CONTROL IN RANDOM TIME

A. *Problem formulation*

Let us consider the adaptive control problem at random time. The objective function determines a sum of costs of control and heredity. Let (Ω, F, P) be a complete probability space. Suppose that $w_1, w_2, \ldots$ are independent $m-$dimensional random vectors on this space, with normal $N(0, I_m)$ distribution, let y_0 be an initial state with distribution $P(dy_0)$, ξ be a system parameters with a'priori distribution $P(d\xi)$ and τ be a random horizon with the same discrete distribution $P(d\tau)$. We assume that all the above mentioned objects are stochastically independent. On (Ω, F, P) we define a family of sub- σ -fields $Y_j = \sigma\{y_i : i = 0,1,\ldots, j\}$ and $Y_j^{\xi} = Y_j \vee \sigma\{\xi\}$.

We will consider the control problem for a system with state equation

$$y_{j+1} = f(y_j, \xi, u_j) + \sigma(y_j) w_{j+1} \tag{1}$$

where $i = 0,1,\ldots, N-1$, $y_i \in R^n$, $f : R^{n \times k \times l} \to R^n$ and $\sigma : R^n \to M(n,m)$, where $M(n,m)$ is the set of $n \times m$ matrices. The functions f, σ are assumed to be continuous in all their variables. A Y_j -measurable vector $u_j \in R^l$ will be called a control action, and $u = (u_0, u_1, \ldots)$ an admissible control. The class of admissible controls is denoted by U . The random variable τ presents the horizon of performance criterion and has discrete distribution $P(\tau = i) = p_i$, where $0 \le p_i \le 1$ and $\sum_{i=0}^{N} p_i = 1$.

To specify the aim of control, we introduce same functions $g : R^n \times R^l \to R$ and $h_i : R^{n \times i} \times R^{l \times i} \to R$ which are continuous and bounded. The objective function has a form

$$J(u) = E\left[\sum_{i=-1}^{\tau-1} g(y_i, u_i) + h_\tau(y_0,...,y_\tau, u_0,...,u_{\tau-1})\right] \quad (2)$$

where $g(y_{-1}, u_{-1}) = 0$ and $u_{-1} = col(0,...,0)$. At any time $0 \le j \le \tau - 1$, which is not a horizon of control, we take the control u_j, and at time τ we do not take the control but only calculate the value of heredity function. In present case the system (1) can be stopped at time $\tau = 0$, then we calculate only value of heredity function.

The aim of optimal control is to minimize the objective function, which is a sum of costs and heredity. Then the task is to find

$$\inf_{u \in U} J(u) \quad (3)$$

and to determine a sequence of admissible control $u^* = (u_0^*,...,u_{\tau-1}^*)$ for which the infimum is attained.

B. Transformation of task with random horizon to task with deterministic horizon

This part of paper presents the transformation of task with random horizon to task with deterministic horizon. Using the definitions of conditional probability and condition expectation the composite costs functional (2) we can present as

$$J(u) = E\left[\sum_{i=-1}^{\tau-1} g(y_i, u_i) + h_\tau(y_0,...,y_\tau, u_0,...,u_{\tau-1})\right]$$
$$= P(\tau = 0)Eh(y_0) + P(\tau = 1)E[g(y_0, u_0) + h_1(y_0, y_1, u_0)] +$$
$$... + P(\tau = N)E\left[\sum_{i=0}^{N-1} g(y_i, u_i) + h_N(y_0,...,y_N, u_0,...,u_{N-1})\right]$$
$$= E\left[\sum_{i=0}^{N-1} g(y_i, u_i)P(\tau > i) + \sum_{i=0}^{N} h_i(y_0,...,y_i, u_0,...,u_{i-1})P(\tau = i)\right]$$

Finally, the above mentioned functional can be presented as

$$J(u) = E\sum_{i=0}^{N} \phi_i(y_0,...,y_i, u_0,...,u_i) \quad (4)$$

where

$$\phi_i(y_0,...,y_i, u_0,...,u_i) = g(y_i, u_i)P(\tau > i) + h_i(y_0,...,y_i, u_0,...,u_{i-1})P(\tau = i) \quad (5)$$

for $j = 0,1,...,N$. From distribution of random horizon τ we see that $P(\tau > N) = 0$. Therefore, we substitute the task of optimal control with random horizon of finite number of events (3) for the task of optimal control with finite horizon

$$\inf_{u \in U} E\sum_{i=0}^{N} \phi_i(y_0,...,y_i, u_0,...,u_i) \quad (6)$$

The expected value of objective function is the same, but the designing of optimal control for task with established horizon is easer. Below we consider the auxiliary (replacement) task (6) to design (made) the optimal control of system (1) with random horizon τ.

Corollary 1. *If $P(\tau = j) = 0$ for $j = 0,1,...,N-1$ and $P(\tau = N) = 1$ we have a classical adaptive control problem with fixed horizon. Suffice in formula (2) to put $\tau = N$.*

Theorem 1. *Suppose, that the functions ϕ_j, $j = 0,1,...,N$ are continuous and bounded, f and ϕ_j are continuously differentiable and $\det(\sigma(y)\sigma^T(y)) \ne 0$ for $y \in R^n$. If u^* is an optimal control of problem (6), then*

$$E\Big\{\nabla_{u_j}\phi_j(y_0,...,y_j, u_0^*,...,u_j^*) + \sum_{i=j+1}^{N}\nabla_{u_j}\phi_i(y_0,...,y_i, u_0^*,...,u_i^*) + \left(\sum_{i=j+1}^{N}\phi_i(y_0,...,y_i, u_0^*,...,u_i^*)\right)\left(y_{j+1} - f(y_j, \xi, u_j^*)\right)^T \cdot \left(\sigma(y_j)\sigma^T(y_j)\right)^{-1}\nabla_{u_j} f(y_j, \xi, u_j^*) \Big| Y_j\Big\} = 0 \quad (7)$$

for $j \in \{0,1,...,N-1\}$, where $E\{X|Y_j\}$ denote the conditional expectation of X given Y_j.

The proof of above theorem can be seen in [4]

III. CONDITIONAL ENTROPY

Let the density of the normal distribution $N(\eta, Q)$ be denoted by

$$\gamma(x-\eta,Q)=\frac{1}{\sqrt{(2\pi)^m\det(Q)}}\exp\left(-\frac{1}{2}(x-\eta)^T Q^{-1}(x-\eta)\right) \quad (8)$$

Definition 1. *In information theory, the entropy is a measure of uncertainty (a measure of unpredictability) associated with a random variable (vector) ξ and calculated as*

$$H(\xi)=-E\ln p(\xi)$$

where $p(\cdot)$ means a density of random variable (vector) ξ.

Let $\Sigma(y)=\sigma(y)\sigma^T(y)$. The density of the joint distribution of $(\xi,y_0,y_1,...,y_i)$ for the dynamic system (1) is given by

$$\mu_i(\xi,y_0,y_1,...,y_i)= \mu_{i-1}(\xi,y_0,y_1,...,y_{i-1})\gamma(y_i-f(\xi,y_{i-1},u_{i-1}),\Sigma(y_{i-1})) \quad (9)$$

where

$$\mu_0(\xi,y_0)=p(\xi)p_0(y_0) \quad (10)$$

and $p(\cdot)$, $p_0(\cdot)$ are the a`priori densities of the random vector ξ and the state vector y_0 respectively. Thus

$$\mu_i(\xi,y_0,y_1,...,y_i)= p(\xi)p_0(y_0)\prod_{j=1}^{i}\gamma(y_j-f(\xi,y_{j-1},u_{j-1}),\Sigma(y_{j-1})) \quad (11)$$

The distribution of ξ conditioned on the $\sigma-$field Y_i is

$$\mu_i(\xi|y_0,y_1,...,y_i)=\frac{\mu_i(\xi,y_0,y_1,...,y_i)}{\int\mu_i(x,y_0,y_1,...,y_i)dx}$$

$$=\frac{p(\xi)\prod_{j=1}^{i}\gamma(y_j-f(\xi,y_{j-1},u_{j-1}),\Sigma(y_{j-1}))}{\int p(\xi)\prod_{j=1}^{i}\gamma(y_j-f(x,y_{j-1},u_{j-1}),\Sigma(y_{j-1}))dx} \quad (12)$$

The conditional entropy of random vector ξ based on states observation $(y_0,y_1,...,y_N)$

$$H(\xi|Y_N)=-E\ln\mu_N(\xi|y_0,y_1,...,y_N) =H(\xi,y_0,y_1,...,y_N)-H(y_0,y_1,...,y_N) \quad (13)$$

We see, that the above entropy is the difference between joint entropy of system and entropy of states system.

Corollary 2. *If the system has a linear form*

$$y_{i+1}=f_1(y_i,u_i)+f_2(y_i,u_i)\xi+\sigma(y_i)w_{i+1} \quad (14)$$

and the random vector ξ has a normal distribution $N(\eta,Q)$ (we assume that f_1,f_2,σ are continuous in all their arguments, bounded and the matrix $\sigma\sigma^T$ is non-singular) then, using the filtration of conditionally normal sequences (see e.g. [14]) the conditional distribution $P(d\xi|Y_N)$ is $N(\eta_N,Q_N)$ where

$$\eta_N=\left[I+Q\sum_{i=0}^{N-1}f_2^T(y_i,u_i)\Sigma^{-1}(y_i)f_2(y_i,u_i)\right]^{-1} \cdot\left[\eta+Q\sum_{i=0}^{N-1}f_2^T(y_i,u_i)\Sigma^{-1}(y_i)(y_{i+1}-f_1(y_i,u_i))\right] \quad (15)$$

and

$$Q_N=\left[I+Q\sum_{i=0}^{N-1}f_2^T(y_i,u_i)\Sigma^{-1}(y_i)f_2(y_i,u_i)\right]^{-1}Q \quad (16)$$

Thus, the condition entropy of random vector ξ based on states observation $(y_0,y_1,...,y_N)$ and controls $(u_0,u_1,...,u_{N-1})$ of system (1) is

$$\begin{aligned}H(\xi|Y_N)&=-\int\ln\gamma(x-\eta_N,Q_N)\gamma(x-\eta_N,Q_N)dx\\&=\frac{1}{2}\int\Big(\ln(2\pi)^m+\ln\det(Q_N)\\&\quad+(x-\eta_N)^T Q_N^{-1}(x-\eta_N)\Big)\gamma(x-\eta_N,Q_N)dx\\&=\frac{1}{2}\int\Big(\ln(2\pi)^m+\ln\det(Q_N)\\&\quad+tr\big(Q_N^{-1}(x-\eta_N)(x-\eta_N)^T\big)\Big)\gamma(x-\eta_N,Q_N)dx\\&=\frac{1}{2}\left(\ln(2\pi)^m+\ln\det(Q_N)+tr\left(Q_N^{-1}Q_N\right)\right)\\&=\frac{1}{2}\left(\ln(2\pi e)^m+\ln\det(Q_N)\right)\end{aligned}$$

Finally

$$H(\xi|Y_N)=\frac{1}{2}\left(\ln(2\pi e)^m -\ln\det\left(Q^{-1}+Q\sum_{i=0}^{N-1}f_2^T(y_i,u_i)\Sigma^{-1}(y_i)f_2(y_i,u_i)\right)\right) \quad (17)$$

Corollary 3. *The conditional covariance matrix Q_j for $j\geq 1$ given (16) can be written in dynamical form*

$$Q_j=\left[QQ_{j-1}^{-1}+f_2^T(y_{j-1},u_{j-1})\Sigma^{-1}(y_{j-1})f_2(y_{j-1},u_{j-1})\right]^{-1}Q$$

with $Q_0=Q$.

IV. OPTIMAL IDENTIFICATION OF LINEAR SYSTEM WITH CONTROL COSTS

Let us consider the case, where the system (1) must be identified at lowest cost. The cost of control (energetic cost) in each step is described by a function $g(y_i, u_i)$ but the joint costs of uncertainty and heredity (e.g. costs of losses associated with instability, no hit to the target,...) are calculated as $\varphi(y_i, H(\xi|Y_i))$. The total cost must be minimized. In case where horizon of control τ is random, the task

$$\inf_{u\in U} E\left\{\sum_{i=0}^{\tau-1} g(y_i,u_i)+\varphi(y_\tau, H(\xi|Y_\tau))\right\} \tag{18}$$

is reduced to replacing task with deterministic horizon

$$\inf_{u\in U} E\left\{\sum_{i=0}^{N-1} g(y_i,u_i)P(\tau>i)+\varphi(y_i,H(\xi|Y_i))P(\tau=i) + \varphi(y_N,H(\xi|Y_N))P(\tau=N)\right\} \tag{19}$$

The necessary condition of optimal control of system (1) for auxiliary task (19) is given below.

Theorem 2. *Suppose, that the functions* g *and* φ *are continuous and bounded,* f *and* φ *are continuously differentiable and* $\det\Sigma(y)\neq 0$ *for* $y\in R^n$. *If* u^* *is an optimal control of problem (19), then*

$$\nabla_{u_j} g(y_j,u_j)P(\tau>j)+E\left\{\sum_{i=j+1}^{N}\nabla_{u_j}\varphi(y_i,H(\xi|Y_i))P(\tau=i)\right.$$
$$+\left(\sum_{i=j+1}^{N-1} g(y_i,u_i)P(\tau>i)+\sum_{i=j+1}^{N}\varphi(y_i,H(\xi|Y_i))P(\tau=i)\right)$$
$$\left.\cdot\left(y_{j+1}-f(y_j,\xi,u_j^*)\right)\Sigma^{-1}(y_j)\nabla_{u_j} f(y_j,\xi,u_j^*)\Big|Y_j\right\}=0 \tag{20}$$

for $j\in\{0,1,\ldots,N-1\}$, *where for* $j<i$

$$\nabla_{u_j}\varphi(y_i,H(\xi|Y_i))=\frac{\partial}{\partial H}\varphi(y_i,H(\xi|Y_i))\cdot\begin{pmatrix}-tr\left(Q_i f_2^T(y_j,u_j)\Sigma^{-1}(y_j)\frac{\partial}{\partial u_j^1} f_2(y_j,u_j)\right)\\ \cdots\cdots\cdots\\ -tr\left(Q_i f_2^T(y_j,u_j)\Sigma^{-1}(y_j)\frac{\partial}{\partial u_j^l} f_2(y_j,u_j)\right)\end{pmatrix}$$

and $u_j = col(u_j^1,\ldots,u_j^l)$.

Corollary 4. *If as the identification measure of unknown parameters is conditional entropy , then for the systems (14) with* $f_2 = const$ *there is no active learning. In this case there is passive learning, the parameters are identified by only observing system states and the conditional entropy does not depend on controls. The effect of active learning occurs when we can influence to component* $f_2(y,u)\xi$ *dependent on system parameters (the conditional entropy depends on the controls).*

Example 1. Let the system be described by a state equation

$$y_{j+1}=y_j+u_j\xi+w_{j+1} \tag{21}$$

where $y_j, u_j\in R$ and random parameter ξ and disturbances w_j have the normal distribution $N(2,3)$ and $N(0,1)$ respectively. The horizon of controls is random and system can be stopped at moments $\{0,1,\ldots,9\}$ thus the controls can be realized at steps $\{0,1,\ldots,8\}$. Let the performance criterion rely on minimizing the sum of controls costs and cost associated with ignorance of parameter ξ, thus the task is to find

$$\inf_{(u_0,u_1,\ldots,u_\tau)} E\left(\alpha\sum_{j=0}^{\tau-1}\|u_j\|^2+\beta H(\xi|Y_\tau)\right) \tag{22}$$

where α,β are the weights of costs. We replace the task (22) by

$$\inf_{(u_0,u_1,\ldots,u_{N-1})} E\left(\sum_{j=0}^{N-1}\left(\alpha\|u_j\|^2 P(\tau>j)+\beta H(\xi|Y_j)P(\tau=j)\right)+\beta H(\xi|Y_N)P(\tau=N)\right)$$

It can be seen, that the objective function is sum of expected costs directly related with controls $\alpha\sum_{j=0}^{N-1}\|u_j\|^2 P(\tau>j)$ and additional costs (losses) of ignorance of system parameter, which is calculated as $\beta\sum_{j=0}^{N} H(\xi|Y_j)P(\tau=j)$. The conditional entropy of ξ is determined as

$$H(\xi|Y_j)=\frac{1}{2}\left(\ln(2\pi)-\ln\left(\frac{1}{3}+\sum_{i=0}^{j-1}u_i^2\right)\right)$$

Let the weights of costs are $\alpha=1$ and $\beta=2$. The table presents the optimal control for different distributions of random horizon τ. In I case the random horizon τ has an uniform distribution $P(\tau=j)=0.1$ for $j=0,1,\ldots,9$ and in

II, III, IV cases has the binomial distributions with probability of success $p = 0.5$, $p = 0.7$, $p = 1$ respectively.

In the present case the task is only to minimize the costs of controls and losses. The table 1 shows dependencies between controls and distribution of random horizon. In case IV $p = 1$ we have got a situation, where the horizon of control is deterministic

$$P(\tau = j) = \begin{cases} 0 & j = 0,1,\ldots,8 \\ 1 & j = 9 \end{cases} \tag{23}$$

TABLE I. THE CONTROL VALUES FOR RANDOM AND FIXED HORIZONS.

j	Case I	Case II	Case III	Case IV
0	-0,814121709	-0,816045733	0,81530348	0,272165477
1	-0,076425343	-0,024357097	0,033712934	0,272165477
2	-0,007104012	-0,008187008	-0,02610353	0,272165477
3	0,006292596	-0,009894085	-0,010232461	0,272165477
4	0,002217472	0,000339139	0,004575022	0,272165477
5	-0,015402668	0,002197519	0,000147496	0,272165477
6	-0,047301927	-0,005335869	-0,004888392	0,272165477
7	-0,081903266	0,028629496	0,000144546	0,272165477
8	0,00146286	0,009621932	0,000957314	0,272165477

It can be seen, in cases I, II, III with random horizon the control values are higher at the beginning and successively decrease in following moments, whereas in case with fixed horizon the control values are evenly (uniformly) distributed.

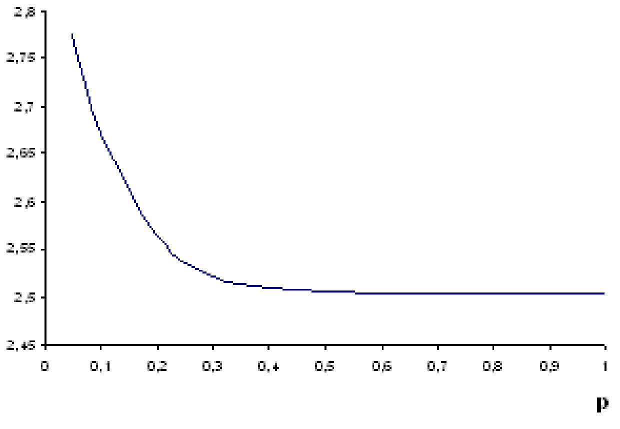

Figure 1. The expected total cost.

The figures 1, 2, 3 present the expected total costs, the expected costs of control and the expected losses (costs) related with ignorance (unknown) of system parameter ξ dependent on probability of success $0 \le p \le 1$ where $\alpha = 1$, $\beta = 2$ and the random horizon τ has a binomial distribution.

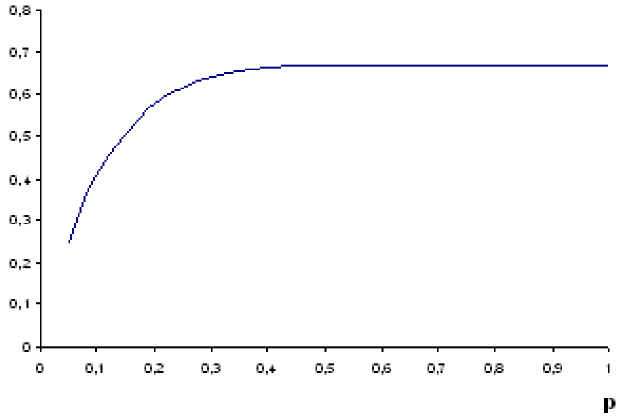

Figure 2. The cost of control.

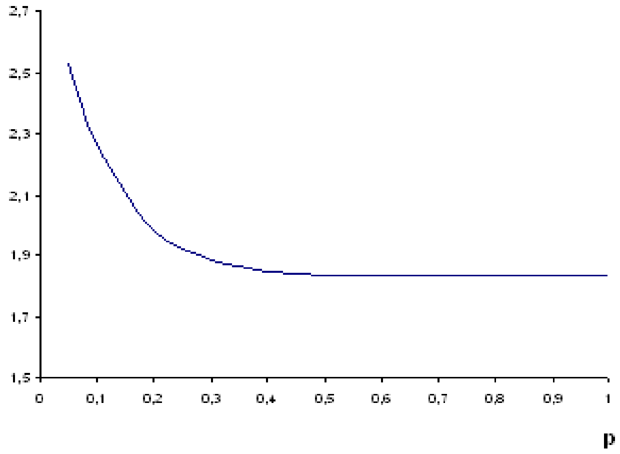

Figure 3. The cost of ignorance of system parameters.

It can be seen, with increasing the probability of success p (it entail that the expected horizon of control increases) the total costs and losses (additionally costs measure by the conditional entropy) are decrease but expected control costs are increase. In this case the greater impact on total cost has the conditional entropy.

Example 2. In the case of construction tasks for the system parameters identification without taking into account the costs of control the optimal control values may be unrealistic, impossible to achieve through the technical systems. For example, to minimize only the conditional entropy without costs of control for a case with random horizon we determine the task

$$\inf_{(u_0,u_1,\ldots,u_\tau)} EH(\xi|Y_\tau)$$

which can be replaced by

$$\inf_{(u_0,u_1,\ldots,u_{N-1})} E\sum_{j=0}^{N-1}\left(H(\xi|Y_j)P(\tau=j)\right)$$

For the case with deterministic horizon the distribution is (23). For the system (21) in both cases the performance criterion is satisfied if the optimal controls $\|u_j\| \to \infty$ for $j = 0,1,\ldots,N-1$.

V. CONCLUSION

In this article, the identification problem of stochastic discrete-time linear system for random horizon was introduced and the optimal control laws were worked out. The random horizon was modeled by random variable at finite number of elementary events. The described problem was reduced to optimal control task with finite horizon. The aims of control for primary and substitute tasks are the same. Additionally, the simple example shows that the optimal control of stochastic system identification in random and fixed time intervals are different. Thus, to design the identification of linear system for random time we can not make directly a task with establish horizon, necessarily we must modify a composite costs function. In the task of identification without costs of control the attention should be given to situations where control may be unlimited.

The extension of described problem can be used for example to system image recognition, diagnosis etc. in random time interval.

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